

ROTA-BAXTER TYPE OPERATORS, REWRITING SYSTEMS AND GRÖBNER-SHIRSHOV BASES

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ABSTRACT. In this paper we apply the methods of rewriting systems and Gröbner-Shirshov bases to give a unified approach to a class of linear operators on associative algebras. These operators resemble the classic Rota-Baxter operator, and they are called *Rota-Baxter type operators*. We characterize a Rota-Baxter type operator by the convergency of a rewriting system associated to the operator. By associating such an operator to a Gröbner-Shirshov basis, we obtain a canonical basis for the free algebras in the category of associative algebras with that operator. This construction include as special cases several previous ones for free objects in similar categories, such as those of Rota-Baxter algebras and Nijenhuis algebras.

CONTENTS

1. Introduction	1
2. Rota-Baxter type operators and rewriting systems	4
2.1. Free operated algebras	4
2.2. Operated polynomial identity algebras	5
2.3. Term-rewriting on free \mathbf{k} -modules	7
2.4. Rota-Baxter term-rewriting	11
3. Rota-Baxter type operators and convergent rewriting systems	15
4. Rota-Baxter type operators and Gröbner-Shirshov basis	23
4.1. CD lemma and the main theorem	23
4.2. Construction of free ϕ -algebra	28
5. Applications to Conjecture 2.37	31
5.1. Monomial order on $\mathfrak{M}(Z)$	31
5.2. Consequences on Rota-Baxter type operators	35
References	38

1. INTRODUCTION

Many years ago, G.-C. Rota [38] posed the question of finding all the algebraic identities that could be satisfied by some linear operator defined on some associative algebra. He wrote:

In a series of papers, I have tried to show that other linear operators satisfying algebraic identities may be of equal importance in studying certain algebraic phenomena, and I have posed the problem of finding all possible algebraic identities that can be satisfied by a linear operator on an algebra. Simple computations show that the possibility are very few, and the problem of classifying all such identities is very probably completely solvable.

Rota was most interested in the following operators arising from analysis, probability and combinatorics:

Endomorphism operator	$d(xy) = d(x)d(y),$
Differential operator	$d(xy) = d(x)y + xd(y),$
Average operator	$P(x)P(y) = P(xP(y)),$
Inverse average operator	$P(x)P(y) = P(P(x)y),$
(Rota-)Baxter operator	$P(x)P(y) = P(xP(y) + P(x)y + \lambda xy),$
of weight λ	where λ is a fixed constant,
Reynolds operator	$P(x)P(y) = P(xP(y) + P(x)y - P(x)P(y)).$

The importance of the endomorphism operator is well-known for the role automorphisms (bijective endomorphisms) play in Galois theory. The differential operator is essential in analysis and its algebraic generalizations led to the development of differential algebra [30, 37], difference algebra [16, 33], and quantum differential operators [36]. The other operators are also important. For example, the Rota-Baxter operator, which originated from probability study [9], is closely related to the classical Yang-Baxter equation, as well as to operads, to combinatorics and, through the Hopf algebra framework of Connes and Kreimer, to the renormalization of quantum field theory [2, 3, 8, 5, 17, 19, 20, 18, 24, 25, 28].

In recent years, new linear operators have emerged from algebraic studies, combinatorics, and physics [15, 26, 32]. Examples are:

Differential operator of weight λ	$d(xy) = d(x)y + xd(y) + \lambda d(x)d(y),$ where λ is a fixed constant,
Nijenhuis operator	$P(x)P(y) = P(xP(y) + P(x)y - P(xy)),$
Leroux's TD operator	$P(x)P(y) = P(xP(y) + P(x)y - xP(1)y).$

These operators in the above two lists can be grouped into two classes. The first two operators in the first list and the first operator in the second satisfy an identity of the form $d(xy) = N(x, y)$, where $N(x, y)$ is some algebraic expression involving x, y , and the operator d . They belong to the class of **differential type operators** (where $N(x, y)$ is required to satisfy some extra conditions), so called because of their resemblance to the differential operator. The remaining operators satisfy an identity of the form $P(x)P(y) = P(B(x, y))$ where $B(x, y)$ is some algebraic expression involving x, y , and the operator P . These belong to the class of **Rota-Baxter type operators** (where $B(x, y)$ is required to satisfy some extra conditions), so called because of their resemblance to the Rota-Baxter operator.

It is interesting to observe that the above operators of differential type share similar properties. Their free objects are constructed in the same way and their studies in general follow parallel paths. The same can be said of Rota-Baxter type operators. After the free objects of Rota-Baxter algebras were constructed in [18, 24], similar constructions have been obtained for free objects for Nijenhuis algebras [31] and for TD algebras [41]. Likewise, the constructions of free commutative Nijenhuis algebras and free commutative TD algebras in [21] are similar to the construction for free commutative Rota-Baxter algebras in [25]. Other instances of similar constructions can be found in [1, 14, 42]. Furthermore, these operators share similar applications:

for example, for the double structures in mathematical physics (especially in the Lie algebra context) [6, 7, 39] and for the splitting of associativity in mathematics [2, 8, 35]. It will be helpful to study these two classes of operators under one theory. On the one hand, we will be able to treat all operators of each type uniformly; for instance in the construction of their free objects. On the other hand, we may discover other operators in these two classes, eventually give a complete list of these operators, and make some progress towards solving Rota's problem.

Following this approach, a systematic investigation on differential type operators is carried out in [27] by studying the operated polynomial identities they satisfy in the framework of operated algebras [23]. These identities are then characterized by means of their rewriting systems [4] and associated Gröbner-Shirshov bases [10, 12, 13].

A conjectured list of Rota-Baxter type operators is provided in [27] based on symbolic computation done in [40]. The study of Rota-Baxter type operators, however, is more challenging than their differential counterpart as can be expected already by comparing integral calculus with differential calculus. Nevertheless the method of Gröbner-Shirshov bases has been successfully applied to the study of Rota-Baxter algebras, differential Rota-Baxter algebras and integro-differential algebras [11, 13, 22]. We show in this paper that the methods of rewriting systems and Gröbner-Shirshov bases apply more generally to Rota-Baxter type algebras as well. As consequences we obtain free objects in these operated algebra categories and verify that the operators in the above-mentioned conjectured list are indeed of Rota-Baxter type.

In Section 2, we associate, to each operated polynomial identity $\phi(x, y) = 0$ of a certain form, a family of rewriting systems on free operated algebras, and define a linear operator satisfying that identity to be of Rota-Baxter type if the rewriting systems have some additional properties. We also restate the conjectured list of 14 Rota-Baxter type algebras announced in [27]. In Section 3, we show that a linear operator is of Rota-Baxter type if and only if the rewriting systems associated with the identity it satisfies are convergent. In Section 4, we introduce the notion of a monomial order on free operated algebras that are compatible with the rewriting systems, which enables us to characterize Rota-Baxter type algebras in terms of Gröbner-Shirshov bases. We show that the linear operator is of Rota-Baxter type precisely when the set of operated polynomials derived from ϕ is a Gröbner-Shirshov basis. When this is the case, we give an explicit construction of a free object in the category of operated algebras satisfying the identity $\phi = 0$. In Section 5, we establish a monomial order needed in Section 4 and verify that the identities in the conjectured list indeed define Rota-Baxter type operators and algebras. Thus, we have achieved a uniform construction of the free objects for all the 14 categories of operated algebras whose defining identities are listed in the conjecture. Our construction generalizes and includes as special cases the known constructions for various operated algebras [11, 13, 14, 18, 31].

Our characterization of Rota-Baxter type operators and identities in terms of Gröbner-Shirshov bases and convergent rewriting systems reveals the power of this general approach. It would be interesting to further apply rewriting system and Gröbner-Shirshov bases techniques to study these operators, with the resolution of Rota's classification problem in mind.

Convention. Throughout this paper, we fix a commutative unitary ring \mathbf{k} . By an algebra we mean an associative (but not necessarily commutative) unitary \mathbf{k} -algebra, unless the contrary is specified. Following common terminology, a non-unitary algebra means one that may not have an identity element.

2. ROTA-BAXTER TYPE OPERATORS AND REWRITING SYSTEMS

In this section, we recall the construction of free operated algebras that gives operated polynomial identity algebras. We also obtain results on term-rewriting systems for free \mathbf{k} -modules. These concepts and results provide us with a framework to define Rota-Baxter type operators for algebras and to give a conjectured list of these operators together with the identity each must satisfy. They also prepare us for our main tasks in later sections.

We begin by reviewing some background on operated algebras.

2.1. Free operated algebras. The construction of free operated algebras was given in [23, 27]. See also [13]. We reproduce that construction here to review the notation.

Definition 2.1. An **operated monoid** (resp. **operated \mathbf{k} -algebra**, resp. **operated \mathbf{k} -module**) is a monoid (resp. \mathbf{k} -algebra, resp. \mathbf{k} -module) U together with a map (resp. \mathbf{k} -linear map, resp. \mathbf{k} -linear map) $P : U \rightarrow U$. A morphism from an operated monoid (resp. \mathbf{k} -algebra, resp. \mathbf{k} -module) U to an operated monoid (resp. \mathbf{k} -algebra, resp. \mathbf{k} -module) V is a monoid (resp. \mathbf{k} -algebra, resp. \mathbf{k} -module) homomorphism $f : U \rightarrow V$ such that $f \circ P = P \circ f$.

Let Y be a set, let $M(Y)$ be the free monoid on Y with identity 1, and let $S(Y)$ be the free semigroup on Y . Let $\lfloor Y \rfloor := \{\lfloor y \rfloor \mid y \in Y\}$ denote a set indexed by Y , but disjoint from Y .

Let X be a given set. We will construct the free operated monoid over X as the limit of a directed system

$$\{\iota_n : \mathfrak{M}_n \rightarrow \mathfrak{M}_{n+1}\}_{n=0}^{\infty}$$

of free monoids $\mathfrak{M}_n := \mathfrak{M}_n(X)$, where the transition morphisms ι_n will be natural embeddings. For this purpose, let $\mathfrak{M}_0 = M(X)$, and let

$$\mathfrak{M}_1 := M(X \cup \lfloor \mathfrak{M}_0 \rfloor).$$

Let ι_0 be the natural embedding $\iota_0 : \mathfrak{M}_0 \hookrightarrow \mathfrak{M}_1$. Assume by induction that for some $n \geq 0$, we have defined, for $0 \leq i \leq n+1$, the free monoids \mathfrak{M}_i with the properties that for $0 \leq i \leq n$, we have $\mathfrak{M}_{i+1} = M(X \cup \lfloor \mathfrak{M}_i \rfloor)$ and natural embeddings $\iota_i : \mathfrak{M}_i \rightarrow \mathfrak{M}_{i+1}$. Let

$$(1) \quad \mathfrak{M}_{n+2} := M(X \cup \lfloor \mathfrak{M}_{n+1} \rfloor).$$

The identity map on X and the embedding ι_n together induce an injection

$$(2) \quad \iota_{n+1} : X \cup \lfloor \mathfrak{M}_n \rfloor \hookrightarrow X \cup \lfloor \mathfrak{M}_{n+1} \rfloor,$$

which, by the functoriality of M , extends to an embedding (still denoted by ι_{n+1}) of free monoids

$$(3) \quad \iota_{n+1} : \mathfrak{M}_{n+1} = M(X \cup \lfloor \mathfrak{M}_n \rfloor) \hookrightarrow M(X \cup \lfloor \mathfrak{M}_{n+1} \rfloor) = \mathfrak{M}_{n+2}.$$

This completes our inductive definition of the directed system. Let

$$\|X\| := \mathfrak{M}(X) := \bigcup_{n \geq 0} \mathfrak{M}_n = \varinjlim \mathfrak{M}_n$$

be the direct limit of the system. Elements of $\mathfrak{M}_n \setminus \mathfrak{M}_{n-1}$ are said to have **depth** n . We note that $\mathfrak{M}(X)$ is a monoid, and by taking direct limit on both sides of $\mathfrak{M}_n = M(X \cup \lfloor \mathfrak{M}_{n-1} \rfloor)$, we obtain

$$(4) \quad \mathfrak{M}(X) = M(X \cup \lfloor \mathfrak{M}(X) \rfloor).$$

Let $\mathbf{k}\mathfrak{M}(X)$ be the (free) \mathbf{k} -module with basis $\mathfrak{M}(X)$. Since the basis is a monoid, the multiplication on $\mathfrak{M}(X)$ can be extended via linearity to turn the \mathbf{k} -module $\mathbf{k}\mathfrak{M}(X)$ into a \mathbf{k} -algebra, which we denote by $\mathbf{k}\mathfrak{M}(X)$. Similarly, we can extend the operator $\lfloor \cdot \rfloor : \mathfrak{M}(X) \rightarrow \mathfrak{M}(X)$, which takes $w \in \mathfrak{M}(X)$ to $\lfloor w \rfloor$, to an operator P on $\mathbf{k}\mathfrak{M}(X)$ by \mathbf{k} -linearity and turn the \mathbf{k} -algebra $\mathbf{k}\mathfrak{M}(X)$

into an operated \mathbf{k} -algebra, which we shall denote by $\mathbf{k}\llbracket X \rrbracket$ (or by abuse, $\mathbf{k}\mathfrak{M}(X)$, since as sets, $\mathfrak{M}(X) = \llbracket X \rrbracket$). If X is a finite set, we may also just list its elements, as in $\mathbf{k}\llbracket x, y \rrbracket$ when $X = \{x, y\}$.

Lemma 2.2. [23] *Let $i_X : X \rightarrow \mathfrak{M}(X)$ and $j_X : \mathfrak{M}(X) \rightarrow \mathbf{k}\llbracket X \rrbracket$ be the natural embeddings. Then, with structures as above,*

- (a) *the triple $(\mathfrak{M}(X), \lfloor \rfloor, i_X)$ is the free operated monoid on X ; and*
- (b) *the triple $(\mathbf{k}\llbracket X \rrbracket, P, j_X \circ i_X)$ is the free operated unitary \mathbf{k} -algebra on X .*

For the rest of this paper, we will use the infix notation $\lfloor r \rfloor$ interchangeably with $P(r)$ for any $r \in R$ where R is an operated algebra with operator P ; for example, when $R = \mathbf{k}\llbracket X \rrbracket$.

Definition 2.3. Elements of $\mathfrak{M}(X)$ are called **bracketed words** or **bracketed monomials in X** . An element $\phi \in \mathbf{k}\llbracket X \rrbracket$ will be called an **operated** or **bracketed polynomial in X with coefficients in \mathbf{k}** , and we will implicitly assume that $\phi \notin \mathbf{k}$, unless otherwise noted. When there is no danger of confusion, we often omit the adjective “bracketed.”

The following notions will be needed for Sections 4 and 5.

Definition 2.4. Let $u \in \mathfrak{M}(X)$, $u \neq 1$. By Eq. (4), we may write u as a product $v_1 \cdots v_k$ uniquely for some k with $v_i \in X \cup \llbracket \mathfrak{M}(X) \rrbracket$ for $1 \leq i \leq k$. We call k the **breadth** of u and denote it by $|u|$. If $u = 1 \in \mathfrak{M}(X)$, we define $|u| = 0$. Alternatively, by combining adjacent factors of $u = v_1 \cdots v_k$ that belong to X into a monomial belonging to $M(X)$ and by inserting $1 \in M(X)$ between two adjacent factors of the form $\lfloor x \rfloor$ where $x \in \mathfrak{M}(X)$, we may write u uniquely in the canonical form

$$(5) \quad u = u_0 \lfloor u_1^* \rfloor u_1 \cdots \lfloor u_r^* \rfloor u_r, \text{ where } u_0, \dots, u_r \in M(X) \text{ and } u_1^*, \dots, u_r^* \in \mathfrak{M}(X).$$

We define the **P -breadth** of u to be r and denote it by $|u|_P$. Note that $|u|_P = 0$ if and only if $u = u_0 \in M(X)$. We further define the **operator degree** $\deg_P(u)$ of a monomial u in $\mathbf{k}\llbracket X \rrbracket$ to be the total number of occurrences of the operator $\lfloor \rfloor$ in the monomial u .

2.2. Operated polynomial identity algebras. Let $k \geq 1$ and $X = \{x_1, x_2, \dots, x_k\}$, let $\phi := \phi_{\lfloor \rfloor}(x_1, \dots, x_k) \in \mathbf{k}\llbracket X \rrbracket$. We call x_1, \dots, x_k the **argument variables** and $\lfloor \rfloor$ the **operator variable**. When $\lfloor \rfloor$ does not appear in ϕ , then $\phi = \phi(x_1, \dots, x_k)$ is just a polynomial and its evaluation can be defined as usual by specializing the argument variables x_1, \dots, x_k . We next define its evaluation in general. Let R be an operated algebra with operator P , and let $r = (r_1, \dots, r_k) \in R^k$. By the universal property of the free operated algebra $\mathbf{k}\llbracket X \rrbracket$, the map $f_r : \{x_1, \dots, x_k\} \rightarrow R$ that sends x_i to r_i induces a unique morphism $\widetilde{f}_r : \mathbf{k}\llbracket x_1, \dots, x_k \rrbracket \rightarrow R$ of operated algebras that extends f_r . Define the **evaluation map** $\phi_{(R,P)} : R^k \rightarrow R$ by:¹

$$(6) \quad \phi_{(R,P)}(r_1, \dots, r_k) := \widetilde{f}_r(\phi_{\lfloor \rfloor}(x_1, \dots, x_k)), \quad r = (r_1, \dots, r_k) \in R^k.$$

We call $\phi_{(R,P)}(r_1, \dots, r_k)$ the **evaluation of the operated polynomial $\phi_{\lfloor \rfloor}(x_1, \dots, x_k)$ at the point (r_1, \dots, r_k) with operator P** . When $\lfloor \rfloor$ does not appear in ϕ , this reduces to the usual notion of evaluation of a polynomial at the point (r_1, \dots, r_k) .

Definition 2.5. Let $\phi \in \mathbf{k}\llbracket x_1, \dots, x_k \rrbracket$. We say that an operated algebra R with operator P is a **ϕ -algebra** and that P is a **ϕ -operator**, if $\phi_{(R,P)}(r_1, \dots, r_k) = 0$ for all $r_1, \dots, r_k \in R$. An **operated polynomial identity algebra** is any ϕ -algebra for some ϕ . If R is a ϕ -algebra, we will say loosely that $\phi = 0$ (or by abuse, ϕ) is an **operated polynomial identity (OPI)** satisfied by R .

¹Later in this paper, we may simplify the notation $\phi_{(R,P)}$ to ϕ_R , or emphasize the multiplication $*$ of R and replace (R, P) by the triple $(R, *, P)$. On the other hand, if there is no possibility of confusion of what R and P should be, we may write $\phi(r_1, \dots, r_k)$ for $\phi_{(R,P)}(r_1, \dots, r_k)$.

Definition 2.6. Given any set Z , and a subset $S \subset \mathbf{k}\llbracket Z \rrbracket$, the **operated ideal** $\text{Id}(S)$ of $\mathbf{k}\llbracket Z \rrbracket$ **generated by** S is the smallest operated ideal containing S .

Given any set Z , let $I_\phi(Z)$ be the operated ideal of $\mathbf{k}\llbracket Z \rrbracket$ generated by the set

$$(7) \quad S_\phi(Z) := \{ \phi_{(\mathbf{k}\llbracket Z \rrbracket, \llbracket \cdot \rrbracket)}(u_1, \dots, u_k) \mid u_1, \dots, u_k \in \mathbf{k}\llbracket Z \rrbracket \}.$$

Then the quotient algebra $\mathbf{k}_\phi\llbracket Z \rrbracket := \mathbf{k}\llbracket Z \rrbracket / I_\phi(Z)$ has a natural structure of a ϕ -algebra.

Proposition 2.7. [4, Theorem 3.5.6] *Let $\phi \in \mathbf{k}\llbracket x_1, \dots, x_k \rrbracket$. Given any set Z , the quotient operated algebra $\mathbf{k}_\phi\llbracket Z \rrbracket$ is the free ϕ -algebra on Z .*

The following definitions are adapted from [13, 27] and supersede those in [27], where the definitions were only preliminary.

Definition 2.8. Let Z be a set, let \star be a symbol not in Z , and let $Z^\star = Z \cup \{\star\}$. By a **\star -bracketed word** (respectively, **\star -bracketed polynomial**) on Z , we mean any word in $\llbracket Z^\star \rrbracket = \mathfrak{M}(Z^\star)$ (respectively, polynomial in $\mathbf{k}\mathfrak{M}(Z^\star)$) with exactly one² occurrence of \star . The set of all \star -bracketed words (respectively, \star -bracketed expressions) on Z is denoted by $\llbracket Z \rrbracket^\star$ or $\mathfrak{M}^\star(Z)$ (respectively, $\mathbf{k}^\star\llbracket Z \rrbracket$).

Let $q \in \llbracket Z \rrbracket^\star$ and $u \in \mathfrak{M}(Z)$. We will use $q|_u$ or $q|_{\star \mapsto u}$ to denote the bracketed word on Z obtained by replacing the symbol \star in q by u . Next, we extend by linearity this notion to elements $s = \sum_i c_i u_i \in \mathbf{k}\mathfrak{M}(Z)$, where $c_i \in \mathbf{k}$ and $u_i \in \mathfrak{M}(Z)$, that is, we define in this case $q|_s$ to be the bracketed expression:

$$q|_s := \sum_i c_i q|_{u_i}.$$

Finally, we extend again by linearity this notation to any $q \in \mathbf{k}^\star\llbracket Z \rrbracket$. Note that in either of these generalized settings, $q|_s$ is usually not a bracketed word but a bracketed polynomial.

With the above notation, we can now describe the operated ideal $\text{Id}(S)$ generated by a subset $S \subseteq \mathbf{k}\llbracket Z \rrbracket$. It is given [13, 27] by

$$(8) \quad \text{Id}(S) = \left\{ \sum_{i=1}^k c_i q_i|_{s_i} \mid k \geq 1 \text{ and } c_i \in \mathbf{k}, q_i \in \mathfrak{M}^\star(Z), s_i \in S \text{ for } 1 \leq i \leq k \right\}.$$

Note that neither the q_i 's nor the s_i 's ($1 \leq i \leq k$) appearing in the above summation expression need be distinct.

Definition 2.9. A bracketed word $u \in \mathfrak{M}(Z)$ is a **subword** of another bracketed word $w \in \mathfrak{M}(Z)$ if $w = q|_u$ for some $q \in \mathfrak{M}^\star(Z)$, where the specific occurrence of u in w is defined by $q|_u$ (that is, by the \star in q). To make this more precise, the set of character positions (when a bracketed word is viewed as a string of characters) occupied by the subword u in the word w under the substitution $q|_u$ is called the **placement of u in w by q** . We denote this placement by the pair (u, q) . A subword u may appear at multiple locations (and hence have distinct placements using distinct q 's) in a bracketed word w .

Example 2.10. Let $Z = \{x, y\}$. Consider placements in the monomial $w = \llbracket xyxy \rrbracket$.

- (a) The subword $u := x$ appears at two locations in w . Their placements are (u, q_1) and (u, q_2) where $q_1 = \star\llbracket x \rrbracket$ and $q_2 = x\llbracket \star \rrbracket$.
- (b) The placement of x in w by $q_1 = \llbracket xy \star y \rrbracket$ is included as a (proper) subset of the placement of xy in w by $q_2 = \llbracket xy \star \rrbracket$. We say (x, q_1) and (x, q_2) are *nested* in w .

²counting multiplicities; thus $q = \star^2$ and $q = \llbracket \star \rrbracket^2$ are not \star -bracketed words.

- (c) The two placements for the subword x are disjoint, as are the two for xy . We say in each case, the two placements are *separated in w* .
- (d) Each of the two placements for xy overlaps partially the unique placement of yx in w . We say they are *intersecting*. The two placements of xx in xxx are also intersecting.

The notions illustrated by these examples will be formally defined later under Definition 3.2.

2.3. Term-rewriting on free \mathbf{k} -modules. In this subsection, we recall some basic definitions and develop new results for term-rewriting systems when they are specialized for free \mathbf{k} -modules with a given basis and satisfy a simple condition. In the next subsection, we apply them to the Rota-Baxter term rewriting systems on operated \mathbf{k} -algebras.

Definition 2.11. Let V be a free \mathbf{k} -module with a given \mathbf{k} -basis W . For $f \in V$, when f is expressed as a unique linear combination of $w \in W$ with coefficients in \mathbf{k} , the **support** $\text{Supp}(f)$ of f is the set consisting of $w \in W$ appearing in f (with non-zero coefficients). Let $f, g \in V$. We use $f \dot{+} g$ to indicate the relation that $\text{Supp}(f) \cap \text{Supp}(g) = \emptyset$. If this is the case, we say $f + g$ is a **direct sum** of f and g , and by abuse,³ we use $f \dot{+} g$ also for the sum $f + g$.

Note $\text{Supp}(0) = \emptyset$ and hence $f \dot{+} 0$ for any $f \in V$. We record the following obvious properties of $\dot{+}$.

Lemma 2.12. Let V be a free \mathbf{k} -module with a \mathbf{k} -basis W . Let $a, b, c \in \mathbf{k}$, let $w \in W$, and let $f, g, h \in V$. Then

- (a) The basis element w is not in $\text{Supp}(f)$ if and only if $w + f = w \dot{+} f$.
- (b) If $f \dot{+} g$, then $af \dot{+} bg$, and if furthermore $f \dot{+} h$, then $af \dot{+} (bg + ch)$.

Definition 2.13. Let V be a free \mathbf{k} -module with a \mathbf{k} -basis W . For $f \in V$ and $w \in \text{Supp}(f)$, let the coefficient of w in f be c_w . We define the w -**complement** of f to be $R_w(f) := f - c_w w \in V$, so that $f = c_w w \dot{+} R_w(f)$.

Definition 2.14. Let V be a free \mathbf{k} -module with a \mathbf{k} -basis W . A **term-rewriting system Π on V with basis W** is a binary relation $\Pi \subseteq W \times V$.

- (a) We say the rewriting system Π is **simple** if $t \dot{+} v$ for all $(t, v) \in \Pi$.
- (b) The image $\pi_1(\Pi)$ of Π under the first projection map $\pi_1 : W \times V \rightarrow W$ will be denoted by T or $T(\Pi)$. A **reducible term** (under Π) is any element $t \in T$.
- (c) For $(t, v) \in \Pi \subseteq T \times V$, we write $t \rightarrow_\Pi v$ and view this as a rewriting rule on V , that is, if $f \in V$, $t \in \text{Supp}(f)$ and $c_t \in \mathbf{k}$ is the coefficient of t in f , then we may apply the rule to f by replacing t with v , resulting in a new element $g := c_t v + R_t(f) \in V$ and say f **reduces to**, or **rewrites to**, g **in one-step** and indicate any such one-step rewriting by $f \rightarrow_\Pi g$, or in more detail, by $f \xrightarrow{(t,v)}_\Pi g$.
- (d) The reflexive transitive closure of \rightarrow_Π (as a binary relation on V) will be denoted by $\xrightarrow{*}_\Pi$ and we say f **reduces to g with respect to Π** if $f \xrightarrow{*}_\Pi g$.
- (e) We say $f \in V$ **reduces to $g \in V$ with respect to Π in n steps** ($n \geq 1$) and denote this by $f \xrightarrow{n}_\Pi g$ if there exist $f_0, f_1, \dots, f_n \in V$ such that $f_i \neq f_{i+1}$ for $i = 0, \dots, n-1$ and $f = f_0 \rightarrow_\Pi f_1 \rightarrow_\Pi \dots \rightarrow_\Pi f_n = g$. An element $f \in V$ is **reducible**⁴ if $f \xrightarrow{n}_\Pi g$ for some $g \in V$ and $n \geq 1$, otherwise we say f is **irreducible** or **in normal form**. We extend

³Whether $\dot{+}$ refers to the relation or the direct sum will always be clear from the context.

⁴In certain context, there may be several term-rewriting systems under discussion, in which case, we use Π -*reducible* for reducible. Similar modification will be used for other terms defined below.

this notation by convention to $f \xrightarrow{0} g$, which means $f = g$, and includes, although not necessarily, the case when f is irreducible.

- (f) Two elements $f, g \in V$ are **joinable** if there exist $p \in V$ such that $f \xrightarrow{*}_{\Pi} p$ and $g \xrightarrow{*}_{\Pi} p$; we denote this by $f \downarrow_{\Pi} g$. If f, g are joinable, the **joinable distance** $d_{\Pi}^V(f, g)$ **between f and g** is the minimum of $m + n$ over all possible p and reductions to p , that is,

$$(9) \quad d_{\Pi}^V(f, g) := \min \{m + n \mid \exists p \in V \text{ such that } f \xrightarrow{m}_{\Pi} p, g \xrightarrow{n}_{\Pi} p, m \geq 0, \text{ and } n \geq 0\}.$$

Definition 2.15. A (term) rewriting system Π on V is called

- (a) **normalizing** if every $f \in V$ reduces to a (not necessarily unique) element in normal form;
- (b) **terminating** if there is no infinite chain of one-step reductions $f_0 \rightarrow_{\Pi} f_1 \rightarrow_{\Pi} f_2 \cdots$;
- (c) **confluent** (resp. **locally confluent**) if every fork (resp. local fork) is joinable; and
- (d) **convergent** if it is both terminating and confluent.

Remark 2.16. The property that a term-rewriting system is simple is a very weak condition. For example, $\Pi = \{(x, y), (y, x)\}$ for the \mathbf{k} -submodule V with basis $W = \{x, y\}$ is simple. Every element of V is reducible, none has a normal form, and Π is neither normalizing nor terminating, but is confluent. For later applications, further restrictions will be imposed so that, for example, the rewriting system is consistent with the algebraic structure. See Definition 2.23.

A well-known result on rewriting systems is Newman's Lemma [4, Lemma 2.7.2].

Lemma 2.17. (Newman) *A terminating rewriting system is confluent if and only if it is locally confluent.*

Proposition 2.18. *Let V be a free \mathbf{k} -module with a \mathbf{k} -basis W and let Π be a simple term-rewriting system on V with respect to W . For any $f, g \in V$, consider the following properties:*

- (a) $f \rightarrow_{\Pi} g$.
- (b) $(f - g) \rightarrow_{\Pi} 0$.
- (c) $(f - g) \xrightarrow{*}_{\Pi} 0$. (equivalently, $(f - g) \downarrow_{\Pi} 0$).
- (d) $f \downarrow_{\Pi} g$.

Then (a) \implies (b) \implies (c) \implies (d).

Proof. (a) \implies (b): Let $f = ct + R_t(f)$ and $g = cv + R_t(g)$, where $(t, v) \in \Pi$. Then since $t \doteq v$, we have $f - g = ct + (-cv) \rightarrow_{\Pi} cv - cv = 0$.

(b) \implies (c): and the equivalence in (c) are obvious.

(c) \implies (d): By hypothesis, $(f - g) \xrightarrow{n}_{\Pi} 0$ for some $n \geq 0$. We prove (c) \implies (d) by induction on n . If $n = 0$, then $f = g$ and hence $f \downarrow_{\Pi} g$. Suppose $n \geq 1$. Then $(f - g) \rightarrow_{\Pi} h$ for some $h \in V$, $h \neq f - g$, and $h \xrightarrow{n-1}_{\Pi} 0$. More specifically, there exist $(t, v) \in \Pi$ and $c_t \in \mathbf{k}$, $c_t \neq 0$, such that $f - g = c_t t + R_t(f - g)$ and $h = c_t v + R_t(f - g)$. Now we may write $f = at + R_t(f)$ and $g = bt + R_t(g)$, where at most one of a, b may be zero. Then $f - g = (a - b)t + (R_t(f) - R_t(g))$ by Lemma 2.12(b), and hence $c_t = a - b$ and $R_t(f - g) = R_t(f) - R_t(g)$. Then rewriting

$$\begin{aligned} h &= c_t t + R_t(f - g) - c_t(t - v) \\ &= (f - g) - (a - b)(t - v) \\ &= (f - a(t - v)) - (g - b(t - v)), \end{aligned}$$

and noting that $h \xrightarrow{\Pi}^{n-1} 0$, we have by induction that $(f - a(t - v)) \downarrow_{\Pi} (g - b(t - v))$. Now

$$f = at \dot{+} R_t(f) \xrightarrow{\Pi}^* av + R_t(f) = f - a(t - v)$$

and similarly $g \xrightarrow{\Pi}^* g - b(t - v)$, so $f \downarrow_{\Pi} g$. \square

There are examples to show that the implications in Proposition 2.18 are all one-way, thus providing a strict hierarchy of binary relations on V .

We next give a more general result than the implication (a) \implies (d) in Proposition 2.18.

Lemma 2.19. *Let V be a free \mathbf{k} -module with a \mathbf{k} -basis W and let Π be a simple term-rewriting system on V with respect to W . Let $f, g, h \in V$. If $f \rightarrow_{\Pi} g$, then $(f + h) \downarrow_{\Pi} (g + h)$.*

Proof. By Definition 2.14(c), $f \rightarrow_{\Pi} g$ means that there exist $(t, v) \in \Pi$ and $0 \neq c \in \mathbf{k}$ such that $f = ct \dot{+} R_t(f)$ and $g = cv + R_t(f)$. Let $h = bt \dot{+} R_t(h)$, where $b \in \mathbf{k}$ (b may be zero). Since Π is simple, we have $t \dot{+} v$. Since $t \dot{+} R_t(f)$, we have $t \dot{+} g$ by Lemma 2.12. Applying Lemma 2.12 again, we get

$$(10) \quad g + h = bt \dot{+} (g + R_t(h)) \xrightarrow{\Pi}^* bv + (g + R_t(h)) = (b + c)v + R_t(f) + R_t(h).$$

On the other hand, we also have

$$(11) \quad f + h = (b + c)t \dot{+} (R_t(f) + R_t(h)) \xrightarrow{\Pi}^* (b + c)v + R_t(f) + R_t(h).$$

By Eqs. (10) and (11), $(f + h) \downarrow_{\Pi} (g + h)$. \square

The following is a key result for applications in later sections.

Theorem 2.20. *Let V be a free \mathbf{k} -module with a \mathbf{k} -basis W and let Π be a simple term-rewriting system on V with respect to W . Consider the following properties on Π :*

(a) Π is confluent, that is, for any $f, g, h \in V$,

$$(f \xrightarrow{\Pi}^* g, f \xrightarrow{\Pi}^* h) \implies g \downarrow_{\Pi} h.$$

(b) For all $f, g, h \in V$,

$$f \downarrow_{\Pi} g, g \downarrow_{\Pi} h \implies f \downarrow_{\Pi} h.$$

(c) For all $f, g, f', g' \in V$,

$$f \downarrow_{\Pi} g, f' \downarrow_{\Pi} g' \implies (f + f') \downarrow_{\Pi} (g + g').$$

(d) For all $r \geq 1$ and $f_1, \dots, f_r, g_1, \dots, g_r \in V$,

$$f_i \downarrow_{\Pi} g_i \quad (1 \leq i \leq r) \text{ and } \sum_{i=1}^r g_i = 0 \implies \left(\sum_{i=1}^r f_i \right) \xrightarrow{\Pi}^* 0.$$

Then (a) \implies (b) \implies (c) \implies (d).

Much more can be said about a simple term-rewriting system. For example the above properties are equivalent.⁵

⁵Further discussions are left out for limit of space, but can be included if the referee or editor prefers.

Proof. (a) \implies (b): Since $f \downarrow_{\Pi} g$, there exists $f' \in V$ such that $f \xrightarrow{*}_{\Pi} f'$ and $g \xrightarrow{*}_{\Pi} f'$. Similarly, since $g \downarrow_{\Pi} h$, there exists $h' \in V$ such that $g \xrightarrow{*}_{\Pi} h'$ and $h \xrightarrow{*}_{\Pi} h'$. Then $(g \xrightarrow{*}_{\Pi} f', g \xrightarrow{*}_{\Pi} h')$ is a fork and since Π is confluent, $f' \downarrow_{\Pi} h'$. Therefore $f \downarrow_{\Pi} h$.

(b) \implies (c): We first consider the special case when $f' = g'$ both of which are denoted by h' . Let $d = d_{\Pi}^{\vee}(f, g)$, and let $m, n \in \mathbb{N}$ be such that $m + n = d$ and by minimality there exist distinct $f_0, f_1, \dots, f_m \in V$ and distinct $g_0, g_1, \dots, g_n \in V$ with

$$f = f_0 \rightarrow_{\Pi} f_1 \rightarrow_{\Pi} \dots \rightarrow_{\Pi} f_m, \quad g = g_0 \rightarrow_{\Pi} g_1 \rightarrow_{\Pi} \dots \rightarrow_{\Pi} g_n,$$

and $f_m = g_n$. If $d = 0$, then $f = g$ and clearly $(f + h') \downarrow_{\Pi} (g + h')$. If $d = 1$, then either $f \rightarrow_{\Pi} g$ or $g \rightarrow_{\Pi} f$ and these cases follow from Lemma 2.19. Suppose now $d = s + 1$ where $s \geq 1$, and suppose by induction that for all $\tilde{f}, \tilde{g} \in V$,

$$\tilde{f} \downarrow_{\Pi} \tilde{g}, d_{\Pi}^{\vee}(\tilde{f}, \tilde{g}) \leq s \implies (\tilde{f} + h') \downarrow_{\Pi} (\tilde{g} + h').$$

Since $d \geq 2$, either $m \geq 1$ or $n \geq 1$ (or both). Without loss of generality, we assume $m \geq 1$. Then $f_1 \downarrow_{\Pi} g$ and $d_{\Pi}^{\vee}(f_1, g) \leq s$. By the induction hypothesis, $(f_1 + h') \downarrow_{\Pi} (g + h')$. It follows by Lemma 2.19 that $(f + h') \downarrow_{\Pi} (f_1 + h')$, and by assumption (b) that $(f + h') \downarrow_{\Pi} (g + h')$. This completes the induction for the special case.

Applying the special case with $h' = f'$, we get $(f + f') \downarrow_{\Pi} (g + f')$, while applying the special case with $h' = g$, we get $(g + f') \downarrow_{\Pi} (g + g')$. Thus, by assumption (b), we have $(f + f') \downarrow_{\Pi} (g + g')$.

(c) \implies (d): An inductive argument shows that, for all $r \geq 1$ and $f_1, \dots, f_r, g_1, \dots, g_r \in V$,

$$f_i \downarrow_{\Pi} g_i \quad (1 \leq i \leq r) \implies \left(\sum_{i=1}^r f_i \right) \downarrow_{\Pi} \left(\sum_{i=1}^r g_i \right).$$

Indeed the case $r = 1$ is trivial and the case $r = 2$ holds by assumption (c). For $r > 2$, by induction, we may assume that $(\sum_{i=1}^{r-1} f_i) \downarrow_{\Pi} (\sum_{i=1}^{r-1} g_i)$ and by the case $r = 2$, $(\sum_{i=1}^{r-1} f_i) + f_r \downarrow_{\Pi} (\sum_{i=1}^{r-1} g_i) + g_r$.

Then (d) follows since $(\sum_{i=1}^r f_i) \downarrow_{\Pi} 0$ implies $(\sum_{i=1}^r f_i) \xrightarrow{*}_{\Pi} 0$. \square

We now introduce a finer concept of confluence.

Definition 2.21. A **local term-fork** is a fork $(ct \rightarrow_{\Pi} cv_1, ct \rightarrow_{\Pi} cv_2)$ where $(t, v_1), (t, v_2) \in \Pi$ and $c \in \mathbf{k}$, $c \neq 0$. The rewriting system Π is **locally term-confluent** if for every local term-fork $(ct \rightarrow_{\Pi} cv_1, ct \rightarrow_{\Pi} cv_2)$, we have $c(v_1 - v_2) \xrightarrow{*}_{\Pi} 0$.⁶

Lemma 2.22. Let V be a free \mathbf{k} -module with a \mathbf{k} -basis W and let Π be a simple term-rewriting system on V . Suppose we have a well-order \leq on W with the property that, for all $(t, v) \in \Pi$, we have $v < t$ in the sense that $w < t$ (that is, $w \leq t$ but $w \neq t$) for all $w \in \text{Supp}(v)$. If Π is locally term-confluent, then it is locally confluent.

Proof. Let $(g \rightarrow_{\Pi} f, g \rightarrow_{\Pi} h)$ be a local fork in V . Then there exist $(t_1, v_1), (t_2, v_2) \in \Pi$ such that $g \xrightarrow{(t_1, v_1)}_{\Pi} f$ and $g \xrightarrow{(t_2, v_2)}_{\Pi} h$.

To prove $f \downarrow_{\Pi} h$, first suppose $t_1 \neq t_2$. Without loss of generality, we may suppose $t_1 > t_2$. Then we may write $g = c_1 t_1 + (c_2 t_2 + r) = c_2 t_2 + (c_1 t_1 + r)$ for some $r \in V$, $c_1, c_2 \in \mathbf{k}$ and $c_1 \neq 0$, $c_2 \neq 0$. Then $f = c_1 v_1 + (c_2 t_2 + r)$ and $h = c_2 v_2 + (c_1 t_1 + r)$. Hence $f - h = c_1(v_1 - t_1) + c_2(t_2 - v_2)$. Since $t_1 > w_1$ for any $w_1 \in \text{Supp}(v_1)$ and $t_1 > t_2 > w_2$ for any $w_2 \in \text{Supp}(v_2)$, $f - h \xrightarrow{(t_1, v_1)}_{\Pi} c_2(t_2 - v_2) \xrightarrow{(t_2, v_2)}_{\Pi} 0$. By Proposition 2.18, we have $f \downarrow_{\Pi} h$.

⁶By Proposition 2.18, this is a stronger condition than $cv_1 \downarrow_{\Pi} cv_2$. On the other hand, this is like Buchberger's S -polynomials reducing to zero for Gröbner basis.

Next, we suppose $t_1 = t_2$. Writing $t := t_1 = t_2$ and $g = ct + R_t(g)$ for some $c \in \mathbf{k}$ and $c \neq 0$, we have $f = cv_1 + R_t(g)$ and $h = cv_2 + R_t(g)$. By hypothesis, the local term-fork $(ct \rightarrow_{\Pi} cv_1, ct \rightarrow_{\Pi} cv_2)$ implies that $f - h = cv_1 - cv_2 \xrightarrow{*}_{\Pi} 0$. By Proposition 2.18, $f \downarrow_{\Pi} h$. \square

2.4. Rota-Baxter term-rewriting. We now apply the general results from the last subsection to the rewriting process from a Rota-Baxter type OPI. Except for those of differential type, which have been considered in [27], and the Reynolds operator, the class of Rota-Baxter type OPI will include the operated identities that interested Rota [38] and were listed in the introduction. Later in the paper, we will use rewriting systems and Gröbner-Shirshov bases to give an explicit construction of the free ϕ -algebra for some Rota-Baxter type OPI ϕ .

We apply the general setup in Section 2.3 to a Rota-Baxter type OPI.

Definition 2.23. Let $\phi(x, y) \in \mathbf{k}\llbracket x, y \rrbracket$ be an OPI of the form $\llbracket x \rrbracket \llbracket y \rrbracket - \llbracket B(x, y) \rrbracket$, where $B(x, y) \in \mathbf{k}\llbracket x, y \rrbracket$.

- (a) A **Rota-Baxter term** or a **ϕ -term** is a bracketed monomial $m \in \llbracket Z \rrbracket$ of the form $q|_{\llbracket u \rrbracket \llbracket v \rrbracket}$, where $u, v \in \llbracket Z \rrbracket$ and $q \in \mathfrak{M}^*(Z)$. Such a triple (q, u, v) is called a **representation of m** . The set of all representations of $m \in \llbracket Z \rrbracket$ will be denoted by Φ_m . If this is the case, we shall call (q, u, v) a **triple**. For $f \in \mathbf{k}\llbracket Z \rrbracket$, if $w = q|_{\llbracket u \rrbracket \llbracket v \rrbracket} \in \text{Supp}(f)$ and the coefficient of w in f is $c \in \mathbf{k}$ with $c \neq 0$, then the w -complement $R_w(f) := f - cw$ of f will also be denoted by $R_{q,u,v}(f)$ and we call it the **(q, u, v) -complement of f** .
- (b) The **Rota-Baxter ϕ -rewriting system (RB ϕ RS)** is the set $\Pi_{\phi}(Z)$ of rewriting rules in the sense of Definition 2.14, when we take $W := \llbracket Z \rrbracket$, $V = \mathbf{k}\llbracket Z \rrbracket$ and

$$(12) \quad \Pi_{\phi} := \Pi_{\phi}(Z) := \left\{ (q|_{\llbracket u \rrbracket \llbracket v \rrbracket}, q|_{\llbracket B(u,v) \rrbracket}) \mid q \in \mathfrak{M}^*(Z), u, v \in \mathfrak{M}(Z) \right\} \subseteq \llbracket Z \rrbracket \times \mathbf{k}\llbracket Z \rrbracket.$$

- (c) We say $f \rightarrow_{\Pi_{\phi}} g$ (in words, f **reduces, or rewrites, to g with respect to Π_{ϕ} in one step**) if there are elements $q \in \mathfrak{M}^*(Z)$, $c \in \mathbf{k}$ ($c \neq 0$), and $u, v \in \mathfrak{M}(Z)$ such that
 - (i) $q|_{\llbracket u \rrbracket \llbracket v \rrbracket} \in \text{Supp}(f)$, and $f = cq|_{\llbracket u \rrbracket \llbracket v \rrbracket} + R_{q,u,v}(f)$; and
 - (ii) $g = cq|_{\llbracket B(u,v) \rrbracket} + R_{q,u,v}(f)$.

In other words, $f \rightarrow_{\Pi_{\phi}} g$ if for some $u, v \in \mathfrak{M}(Z)$, g is obtained from f by replacing *exactly once* a subword $\llbracket u \rrbracket \llbracket v \rrbracket$ in *one* monomial $t \in \text{Supp}(f)$ by $\llbracket B(u, v) \rrbracket$. When we want to emphasize the parameters involved in this reduction, we write $f \xrightarrow{q,u,v}_{\Pi_{\phi}} g$.

Then Eq. (12) can be expressed as

$$(13) \quad \Pi_{\phi} := \Pi_{\phi}(Z) := \left\{ q|_{\llbracket u \rrbracket \llbracket v \rrbracket} \xrightarrow{q,u,v}_{\Pi} q|_{\llbracket B(u,v) \rrbracket} \mid q \in \mathfrak{M}^*(Z), u, v \in \mathfrak{M}(Z) \right\}.$$

In the following, we shall also denote $\rightarrow_{\Pi_{\phi}}$ (resp. $\xrightarrow{q,u,v}_{\Pi_{\phi}}$, $\xrightarrow{*}_{\Pi_{\phi}}$) simply by \rightarrow_{ϕ} (resp. $\xrightarrow{q,u,v}_{\phi}$, $\xrightarrow{*}_{\phi}$). We also abbreviate Eq. (13) by

$$(14) \quad \Pi_{\phi} := \Pi_{\phi}(Z) := \left\{ \llbracket u \rrbracket \llbracket v \rrbracket \rightarrow_{\phi} \llbracket B(u, v) \rrbracket \mid u, v \in \mathfrak{M}(Z) \right\}.$$

Definition 2.24. Let W be a subset of $\llbracket Z \rrbracket$ and let V be the free \mathbf{k} -submodule of $\mathbf{k}\llbracket Z \rrbracket$ with basis W . We say the rewriting system \rightarrow_{ϕ} on $\mathbf{k}\llbracket Z \rrbracket$ **restricts to** a rewriting system \rightarrow_{Π} on V with basis W if for all $f \in V$, $u, v \in \llbracket Z \rrbracket$ and $q \in \mathfrak{M}^*(Z)$ such that $q|_{\llbracket u \rrbracket \llbracket v \rrbracket}$ is in $\text{Supp}(f)$, we have $f \xrightarrow{q,u,v}_{\phi} g$ implies $g \in V$.

Remark 2.25. A sufficient condition that \rightarrow_{ϕ} restricts to \rightarrow_{Π} is when $q|_{\llbracket B(u,v) \rrbracket} \in V$ whenever $q|_{\llbracket u \rrbracket \llbracket v \rrbracket} \in W$.

Lemma 2.26. *For all $u, v \in \mathfrak{M}(Z)$, $E \in \mathbf{k}\llbracket Z \rrbracket$, and $q \in \mathfrak{M}^*(Z)$, we have $q|_{[u][v]} \dot{+} q|_{[E]}$.*

Proof. First note that left and right multiplication on $\mathfrak{M}^*(Z)$ by an element of $\mathfrak{M}(Z)$ is injective, as is the map sending $q \in \mathfrak{M}^*(Z)$ to $[q]$.

By Lemma 2.12, we only need to prove the lemma for monomials E for which we apply induction on the depth of q . If the depth of q is 0, then $q = q_1 \star q_2$ for $q_1, q_2 \in M(Z)$. Suppose there is $c \in \mathbf{k}$ such that $q|_{[u][v]} = cq|_{[E]}$. Then we obtain $[u][v] = c[E]$. This is a contradiction since the two sides have different breadths. Thus we have $q|_{[u][v]} \dot{+} q|_{[E]}$. Assume that the lemma has been proved for q with depth less or equal to $n \geq 1$ and consider q with depth $n + 1$. Then we have $q = q_1[q']q_2$ with $q_1, q_2 \in \mathfrak{M}(Z)$ and $q' \in \mathfrak{M}^*(Z)$ with depth n . Thus from $q|_{[u][v]} = cq|_{[E]}$ with $c \in \mathbf{k}$, we obtain

$$q_1[q'|_{[u][v]}]q_2 = cq_1[q'|_{[E]}]q_2.$$

From this we obtain $q'|_{[u][v]} = cq'|_{[E]}$. This contradicts the induction hypothesis. Thus we have $q|_{[u][v]} \dot{+} q|_{[E]}$, completing the induction. \square

Corollary 2.27. *The RB ϕ RS Π_ϕ is a simple rewriting system in the sense of Definition 2.14.*

We recall here some basic notions of rewriting systems that we specialize to $\Pi_\phi(Z)$ for any set Z (including the case $Z = X = \{x, y\}$).

Definition 2.28. We say a bracketed polynomial $f \in \mathbf{k}\llbracket Z \rrbracket$ is **ϕ -irreducible** (that is, irreducible with respect to $\Pi_\phi(Z)$) or is in **(Rota-Baxter) normal form** (RBNF) if no monomial of f has $[u][v]$ as a subword for any two monomials $u, v \in \llbracket Z \rrbracket$; otherwise, we say f is **ϕ -reducible**. Equivalently, f is ϕ -reducible if there exists $g \in \mathbf{k}\llbracket Z \rrbracket$, $g \neq f$, such that $f \rightarrow_\phi g$. A bracketed polynomial $g \in \mathbf{k}\llbracket Z \rrbracket$ is said to be a **normal ϕ -form** for f if g is in RBNF and $f \xrightarrow{*}_\phi g$.

In particular, if f is a *monomial*, then it is in RBNF if and only if there do not exist $q \in \mathfrak{M}^*(Z)$ and $u, v \in \llbracket Z \rrbracket$ such that $f = q|_{[u][v]}$. Let $\mathfrak{R}(Z)$ denote the set of *monomials* of $\llbracket Z \rrbracket$ in RBNF.

For a set X , the monomials in X in RBNF are called Rota-Baxter words (RBW) in X in [18]. They are so named since they form a canonical basis of the free Rota-Baxter algebra on X . We will see later that they also form a canonical basis for some other Rota-Baxter type algebras. As can be seen by removing all appearances of the superfluous monoid unit 1 from a representation given by Eq. (5), every monomial $\mathfrak{x} \in \mathfrak{M}(X)$ in RBNF that is not the monoid unit 1 has a unique decomposition of the form

$$(15) \quad \mathfrak{x} = \mathfrak{x}_1 \cdots \mathfrak{x}_k,$$

where the \mathfrak{x}_i for $1 \leq i \leq k$ alternate to belong to either $S(X)$ or $[\mathfrak{R}(X)]$.

Definition 2.29. An expression $B \in \mathbf{k}\llbracket X \rrbracket$ is **totally linear** in X if every variables $x \in X$ appears exactly once in every monomial of B , when counted with multiplicity in repeated multiplications.

Example 2.30. Let $X = \{x, y\}$. The expression $x[y] + [x][y] + xy$ is totally linear in X , but the monomials $[x]$, $x^2[y]$ and $x[y]^2$ are not.

The following definition is extracted from key properties of Rota-Baxter operators.

Definition 2.31. An expression $\phi \in \mathbf{k}\llbracket x, y \rrbracket$ (more correctly, the OPI $\phi = 0$) is a **Rota-Baxter type OPI** if ϕ has the form $[x][y] - [B(x, y)]$ for some $B(x, y) \in \mathbf{k}\llbracket x, y \rrbracket$ and if the following four conditions are satisfied:

- (a) $B(x, y)$ is **totally linear** in x, y ;

- (b) $B(x, y)$ is in RBNF;
- (c) For every set Z , the rewriting system $\Pi_\phi(Z)$ in Eq. (14) is terminating;
- (d) For every set Z and for all $u, v, w \in \mathfrak{M}(Z)$, the expression $B(B(u, v), w) - B(u, B(v, w))$ is ϕ -reducible to zero.

If $\phi := [x][y] - [B(x, y)]$ is of Rota-Baxter type, then we say the expression $B(x, y)$ and the defining operator $P = []$ of a ϕ -algebra R are **of Rota-Baxter type**, too. By a **Rota-Baxter type algebra**, we mean some ϕ -algebra R where ϕ is some expression in $\mathbf{k}\langle x, y \rangle$ of Rota-Baxter type.

Example 2.32. [27] Let $B(x, y) := x[y]$. Then $\phi = 0$ is the OPI defining the average operator and it is of Rota-Baxter type. As will be shown in Theorem 5.10, the identities defining a Rota-Baxter operator and that defining a Nijenhuis operator are OPIs of Rota-Baxter type.

Example 2.33. The expression $B(x, y) := y[x]$ is not of Rota-Baxter type. This is because in $\mathbf{k}\langle u, v, w \rangle$, the operated polynomial

$$\begin{aligned} B(B(u, v), w) - B(u, B(v, w)) &= w[B(u, v)] - B(v, w)[u] \\ &= w[v[u]] - w[v][u] \\ &\rightarrow_\phi w[v[u]] - w[u[v]] \end{aligned}$$

is in RBNF but is non-zero, and there is no other sequence of reduction for $w[v[u]] - w[u[v]]$.

Remark 2.34. Condition (a) in Definition 2.31 is imposed since we are considering linear operators. Conditions (b) and (c) are needed to avoid obvious infinite rewriting under $\Pi_\phi(Z)$ though their relationship is still quite mysterious. Condition (d) is to ensure compatibility with the associative law for products of the form $[a][b][c]$ where $a, b, c \in R$ for a ϕ -algebra R .

The next proposition shows that a ϕ -algebra is also a ψ -algebra if $\psi \xrightarrow{*}_\phi 0$. In the next two propositions, for clarity, we spell out the algebra and ϕ -algebra structure explicitly when needed. For example, if $B(x, y) \in \mathbf{k}\langle x, y \rangle$, then $B_{(R, *, P)}$ refers to the set map from $(R, *, P)^2 \rightarrow (R, *, P)$ (see Footnote 1).

Proposition 2.35. Let $R = (R, *, P)$ be a ϕ -algebra, where $\phi := [x][y] - [B(x, y)]$. Then for any set Z , any finite number of distinct symbols $z_1, \dots, z_k \in Z$, and any operated polynomial $\psi = \psi(z_1, \dots, z_k)$ in $\mathbf{k}\langle Z \rangle$ such that $\psi \xrightarrow{*}_\phi 0$, the ϕ -algebra R is also a ψ -algebra.

Proof. It suffices to show that if $\psi \rightarrow_\phi \psi'$, then $\psi_R(r_1, \dots, r_k) = \psi'_R(r_1, \dots, r_k)$ for all $r_1, \dots, r_k \in R$. Let $\psi \rightarrow_\phi \psi'$. Then there exist $q \in \mathfrak{M}^*(Z)$, $c \in \mathbf{k}$ ($c \neq 0$), and $u, v \in \mathfrak{M}(Z)$ such that

- (a) $q|_{[u][v]}$ is a monomial of ψ , which has c as its coefficient.
- (b) $\psi' = \psi - cq|_{([u][v] - [B(u, v)])}$.

By increasing k if necessary, we may assume $u = u(z_1, \dots, z_k)$ and $v = v(z_1, \dots, z_k)$ are in $\langle z_1, \dots, z_k \rangle$ and $q \in \mathfrak{M}^*(z_1, \dots, z_k)$. Then for any $r_1, \dots, r_k \in R$, the elements $a = u_R(r_1, \dots, r_k)$ and $b = v_R(r_1, \dots, r_k)$ are in R . Since R is a ϕ -algebra, $[a][b] - [B_R(a, b)] = \phi_R(a, b) = 0$, and hence

$$\psi'_R(r_1, \dots, r_k) = \psi_R(r_1, \dots, r_k) - cq_R(r_1, \dots, r_k)|_{([a][b] - [B(a, b)])} = \psi_R(r_1, \dots, r_k).$$

This completes the proof. \square

For a Rota-Baxter algebra R , with multiplication $*$ and Rota-Baxter operator P , it is common to endow R with another multiplication in terms of the defining operator identity. This double algebra structure plays important roles in the splitting of associativity in algebras such as the

dendriform algebra and more generally successors of operads[8, 34, 35], and in integrable systems in the Lie algebra context [6, 7, 39]. We describe this double structure below for the more general Rota-Baxter type algebras (for the case of Rota-Baxter operator, see [24, § 1.1.17]).

Proposition 2.36. *Let $\phi \in \mathbf{k}\llbracket x, y \rrbracket$ be of Rota-Baxter type and suppose $\phi = \llbracket x \rrbracket \llbracket y \rrbracket - \llbracket B(x, y) \rrbracket$. Let $(R, *, Q)$ be a ϕ -algebra. Define a second multiplication $*_\phi$ by*

$$r_1 *_\phi r_2 := B_{(R, *, Q)}(r_1, r_2), \quad \text{for all } r_1, r_2 \in R.$$

Then

- (a) *The pair $(R, *_\phi)$ is a nonunitary \mathbf{k} -algebra.*
- (b) *If $B(x, y)$ does not involve $\llbracket 1 \rrbracket \in \mathbf{k}\llbracket x, y \rrbracket$, then the triple $(R, *_\phi, Q)$ is a nonunitary ϕ -algebra.*

Proof. For clarity, we now use P to denote the operator $\llbracket \cdot \rrbracket$ for $\mathbf{k}\llbracket x, y \rrbracket$, so for example $P(1) = \llbracket 1 \rrbracket$, P^i is the i -fold iteration of P , and P^0 is the identity operator. We observe that since $B(x, y) \in \mathbf{k}\llbracket x, y \rrbracket$ is totally linear in x, y and is in RBNF, we can write

$$(16) \quad B(x, y) = \sum_{j \in J} a_j B_j(x, y),$$

where J is a finite set, for $j \in J$, $B_j(x, y)$ are distinct, totally-linear monomials in RBNF and do not involving $P(1)$ in $\mathfrak{M}(x, y)$, and $a_j \in \mathbf{k}$ and $a_j \neq 0$. Hence $B_j(x, y)$ has one of two forms

$$(17) \quad \text{either } B_j(x, y) = P^{k_j}(P^{m_j}(x)P^{n_j}(y)) \quad \text{or} \quad B_j(x, y) = P^{k_j}(P^{n_j}(y)P^{m_j}(x))$$

with integers $k_j, m_j, n_j \geq 0$ and $m_j n_j = 0$.⁷

(a) Applying Lemma 2.2 with X set to $Z := \{u, v, w\}$, let $(R', *, P')$ be the free operated algebra $\mathbf{k}\llbracket Z \rrbracket = \mathbf{k}\llbracket u, v, w \rrbracket$ on the set Z . The operated polynomial

$$(18) \quad \psi := B_{(R', *, P')}(B_{(R', *, P')}(u, v), w) - B_{(R', *, P')}(u, B_{(R', *, P')}(v, w))$$

of R' is ϕ -reducible to zero. Since $(R, *, Q)$ is a ϕ -algebra, the associativity of $*_\phi$ in $(R, *_\phi)$ holds if and only if $(R, *, Q)$ is a ψ -algebra, which is the case by Proposition 2.35. Similarly, consider the operated polynomials

$$(19) \quad \psi_1 := B_{(R', *, P')}(u, v + w) - B_{(R', *, P')}(u, v) - B_{(R', *, P')}(u, w),$$

$$(20) \quad \psi_2 := B_{(R', *, P')}(v + w, u) - B_{(R', *, P')}(v, u) - B_{(R', *, P')}(w, u).$$

By Eq. (18), the linearity of P' , and the distributive laws of $*$, we get $\psi_1 = \psi_2 = 0$ and hence $(R, *_\phi)$ satisfies the left and right distributive laws. Thus $(R, *_\phi)$ is a nonunitary \mathbf{k} -algebra.

(b) To prove that $(R, *_\phi, Q)$ is a ϕ -algebra, we must show that $\phi_{(R, *_\phi, Q)}(r_1, r_2) = 0$ for all $r_1, r_2 \in R$. We partition the index set J from Eq. (16) accordingly into two disjoint sets J_1, J_2 , where $B_j \in J_1$ has the first form from Eq. (17), and $B_j \in J_2$ has the second form.

⁷The careful reader will note that in the proof of (b), we never make use of the property that $m_j n_j = 0$. Thus the operated polynomial identity $\phi_{(R, *_\phi, Q)}(r_1, r_2) = 0$ holds under a much weaker assumption, requiring only that $B(x, y)$ be totally linear but not necessarily in RBNF. However, considering Examples 2.32 and 2.33, we want to emphasize the importance that $B(x, y)$ be of Rota-Baxter type to begin with for (a) to hold, that is, for $*_\phi$ to be associative.

For any $r_1, r_2 \in R$, we have

$$\begin{aligned}
 Q(r_1) *_{\phi} Q(r_2) &= B_{(R, *, Q)}(Q(r_1), Q(r_2)) \\
 &= \sum_{j \in J_1} a_j Q^{k_j} (Q^{m_j+1}(r_1) * Q^{n_j+1}(r_2)) + \sum_{j \in J_2} a_j Q^{k_j} (Q^{n_j+1}(r_2) * Q^{m_j+1}(r_1)) \\
 &= \sum_{j \in J_1} a_j Q^{k_j} (Q(Q^{m_j}(r_1)) * Q(Q^{n_j}(r_2))) + \sum_{j \in J_2} a_j Q^{k_j} (Q(Q^{n_j}(r_2)) * Q(Q^{m_j}(r_1))) \\
 &= \sum_{j \in J_1} a_j Q^{k_j} (Q(B_{(R, *, Q)}(Q^{m_j}(r_1), Q^{n_j}(r_2)))) + \sum_{j \in J_2} a_j Q^{k_j} (Q(B_{(R, *, Q)}(Q^{n_j}(r_2), Q^{m_j}(r_1)))) \\
 &= \sum_{j \in J_1} a_j Q^{k_j+1} (Q^{m_j}(r_1) *_{\phi} Q^{n_j}(r_2)) + \sum_{j \in J_2} a_j Q^{k_j+1} (Q^{n_j}(r_2) *_{\phi} Q^{m_j}(r_1)) \\
 &= Q \left(\sum_{j \in J_1} a_j Q^{k_j} (Q^{m_j}(r_1) *_{\phi} Q^{n_j}(r_2)) + \sum_{j \in J_2} a_j Q^{k_j} (Q^{n_j}(r_2) *_{\phi} Q^{m_j}(r_1)) \right) \\
 &= Q(B_{(R, *, \phi, Q)}(r_1, r_2)).
 \end{aligned}$$

Thus $\phi_{(R, *, \phi, Q)}(r_1, r_2) = 0$ for all $r_1, r_2 \in R$ and $(R, *, \phi, Q)$ is a non-unitary ϕ -algebra (of Rota-Baxter type). \square

We recall the following conjecture on Rota-Baxter type operators as a case of Rota's problem.

Conjecture 2.37. (Classification of Rota-Baxter Type Operators)[27] *For any $c, \lambda \in \mathbf{k}$, the operated polynomial $\phi := \lfloor x \rfloor \lfloor y \rfloor - \lfloor B(x, y) \rfloor$, where $B(x, y)$ is taken from the list below, is of Rota-Baxter type. Moreover, any OPI ϕ of Rota-Baxter type is necessarily defined as above by a $B(x, y)$ from among this list (new types are underlined).*

- (a) $x \lfloor y \rfloor$ (average operator),
- (b) $\lfloor x \rfloor y$ (inverse average operator),
- (c) $\underline{x \lfloor y \rfloor + y \lfloor x \rfloor}$,
- (d) $\underline{\lfloor x \rfloor y + \lfloor y \rfloor x}$,
- (e) $x \lfloor y \rfloor + \lfloor x \rfloor y - \lfloor xy \rfloor$ (Nijenhuis operator),
- (f) $x \lfloor y \rfloor + \lfloor x \rfloor y + \lambda xy$ (Rota-Baxter operator of weight λ),
- (g) $\underline{x \lfloor y \rfloor - x \lfloor 1 \rfloor y + \lambda xy}$,
- (h) $\underline{\lfloor x \rfloor y - x \lfloor 1 \rfloor y + \lambda xy}$,
- (i) $\underline{x \lfloor y \rfloor + \lfloor x \rfloor y - x \lfloor 1 \rfloor y + \lambda xy}$ (generalized Leroux TD operator with weight λ),
- (j) $\underline{x \lfloor y \rfloor + \lfloor x \rfloor y - xy \lfloor 1 \rfloor - x \lfloor 1 \rfloor y + \lambda xy}$,
- (k) $\underline{x \lfloor y \rfloor + \lfloor x \rfloor y - x \lfloor 1 \rfloor y - \lfloor xy \rfloor + \lambda xy}$,
- (l) $\underline{x \lfloor y \rfloor + \lfloor x \rfloor y - x \lfloor 1 \rfloor y - \lfloor 1 \rfloor xy + \lambda xy}$,
- (m) $\underline{cx \lfloor 1 \rfloor y + \lambda xy}$ (generalized endomorphisms),
- (n) $\underline{cy \lfloor 1 \rfloor x + \lambda yx}$ (generalized antimorphisms).

3. ROTA-BAXTER TYPE OPERATORS AND CONVERGENT REWRITING SYSTEMS

In this section, we shall establish the close relationship between a Rota-Baxter type OPI ϕ and the convergence of its rewriting systems $\Pi_{\phi}(Z)$ on $\mathbf{k} \llbracket Z \rrbracket$ for sets Z in the presence of a monomial

order that is compatible with Π_ϕ . It would be interesting to explore how the latter condition can be weakened or removed.

Definition 3.1. Let Z be a set. For distinct symbols \star_1, \star_2 not in Z , let $Z^{\star_1, \star_2} := Z \cup \{\star_1, \star_2\}$. We define a (\star_1, \star_2) -**bracketed word on Z** to be a bracketed word in $\llbracket Z^{\star_1, \star_2} \rrbracket = \mathfrak{M}(Z^{\star_1, \star_2})$ with exactly one occurrence of \star_1 and exactly one occurrence of \star_2 , each counted with multiplicity. The set of (\star_1, \star_2) -bracketed words on Z is denoted by either $\llbracket Z \rrbracket^{\star_1, \star_2}$ or $\mathfrak{M}^{\star_1, \star_2}(Z)$. For $q \in \llbracket Z \rrbracket^{\star_1, \star_2}$ and $u_1, u_2 \in \mathbf{k} \llbracket Z \rrbracket$, we define

$$(21) \quad q|_{u_1, u_2} = q|_{\star_1 \mapsto u_1, \star_2 \mapsto u_2},$$

to be the bracketed word in $\mathfrak{M}(Z)$ obtained by replacing the symbol \star_1 in q by u_1 and replacing the symbol \star_2 in q by u_2 , simultaneously. A (u_1, u_2) -**bracketed word on Z** is a word of the form Eq. (21) for some $q \in \llbracket Z \rrbracket^{\star_1, \star_2}$.

A (u_1, u_2) -bracketed word on Z can also be recursively defined by

$$(22) \quad q|_{u_1, u_2} := (q^{\star_1}|_{u_1})|_{u_2},$$

where q^{\star_1} is q when q is regarded as a \star_1 -bracketed word on the set $Z \cup \{\star_2\}$. Then $q^{\star_1}|_{u_1}$ is in $\mathfrak{M}^{\star_2}(Z)$ and hence Eq. (22) is well-defined. Similarly, treating q first as a \star_2 -bracketed word q^{\star_2} on the set $Z \cup \{\star_1\}$, we have

$$(23) \quad q|_{u_1, u_2} := (q^{\star_2}|_{u_2})|_{u_1}.$$

We describe the relative location of two bracketed subwords, or more precisely, their placements (Definition 2.9), in a bracketed word. See Example 2.10 for motivation and [43] for details.

Definition 3.2. Let $w, u_1, u_2 \in \llbracket Z \rrbracket$ and $q_1, q_2 \in \mathfrak{M}^{\star}(Z)$ be such that

$$(24) \quad q_1|_{u_1} = w = q_2|_{u_2}.$$

The two placements (u_1, q_1) and (u_2, q_2) are said to be

- (a) **separated** if there exist an element q in $\mathfrak{M}^{\star_1, \star_2}(Z)$ and $a, b \in \llbracket Z \rrbracket$ such that $q_1|_{\star_1} = q|_{\star_1, b}$, $q_2|_{\star_2} = q|_{a, \star_2}$ and $w = q|_{a, b}$;
- (b) **nested** if there exists an element q in $\mathfrak{M}^{\star}(Z)$ such that either $q_2 = q_1|_q$ or $q_1 = q_2|_q$;
- (c) **intersecting** if there exist an element q in $\mathfrak{M}^{\star}(Z)$ and elements a, b, c in $\mathfrak{M}(Z) \setminus \{1\}$ such that $w = q|_{abc}$ and either
 - (i) $q_1 = q|_{\star_1 c}, q_2 = q|_{a \star_2}$; or
 - (ii) $q_1 = q|_{a \star_1}, q_2 = q|_{\star_2 c}$.

Remark 3.3. The defining conditions in Definition 3.2 apparently are properties of q_1 and q_2 that have nothing to do with u_1 and u_2 , but actually they constrain the placements of u_1 and u_2 to be separated, nested or intersecting for any two subwords u_1, u_2 satisfying Eq. (24). The conditions are thus stronger than the requirement for a particular pair of u_1, u_2 . To illustrate this, we view q_1, q_2 as strings and write $q_1 = \ell_1 \star_1 r_1$ and $q_2 = \ell_2 \star_2 r_2$.

- (a) Condition (a) implies $q|_{\star_1, b} = \ell_1 \star_1 r_1$ and $q|_{a, \star_2} = \ell_2 \star_2 r_2$. Hence $w = \ell_1 a r_1 = \ell_2 b r_2$. By Eq. (24), we must have $u_1 = a$ and $u_2 = b$. Since q is of the form either $\ell \star_1 m \star_2 r$ or $\ell \star_2 m \star_1 r$, the placements u_1 and u_2 in w are separated. Note, however, that even if u_1 is a subword of u_2 (including equality), their placements may still be separated.

- (b) Suppose $q_2 = q_1|_q$ in Condition (b) is satisfied and we write $q = \ell \star r$. Then $\ell_2 \star r_2 = \ell_1 \ell \star rr_1$ so that $\ell_2 = \ell_1 \ell$ and $r_2 = rr_1$. This means q_1 and q_2 share an initial string ℓ_1 and an ending string r_1 ; replacing the inner \star -word q by \star collapses q_2 to q_1 . From Eq. (24), $\ell_1 u_1 r_1 = w = \ell_2 u_2 r_2 = \ell_1 \ell u_2 r r_1$ and hence

$$(25) \quad u_1 = q|_{u_2}.$$

Thus u_2 is a bracketed subword of u_1 . Note that a special case of being nested is when $(u_1, q_1) = (u_2, q_2)$ with $q = \star$.

- (c) Suppose (i) under Condition (c) holds. Then we have $w = q_1|_{u_1} = (q|_{\star c})|_{u_1} = q|_{u_1 c}$ and $w = q_2|_{u_2} = (q|_{a \star})|_{u_2} = q|_{a u_2}$. Then we have $u_1 c = a u_2 = a b c$. Thus $u_1 = a b$, $u_2 = b c$ and they have a nontrivial intersection b . Note that the existence of q satisfying all the conditions is crucial. Example: Let $b \neq 1$, $y = a b c$, $w = x a b y b c z$, $u_1 = a b$, $u_2 = b c$, $q_1 = x \star y b c z$, and $q_2 = x a b y \star z$. Then $q_1|_{u_1} = w = q_2|_{u_2}$. The placements (u_1, q_1) and (u_2, q_2) do not overlap, even though $a b c = y$ is a subword of w —it occurs at the wrong place.

Example 3.4. Let $w = x[x]x$, $u_1 = x[x]$ and $u_2 = x$. Then u_1 is a bracketed subword of w with placement (u_1, q_1) where $q_1 = \star x$. Also u_2 is a bracketed subword of w in three locations with placements (u_2, q_{21}) , (u_2, q_{22}) , and (u_2, q_{23}) , where $q_{21} = \star[x]x$, $q_{22} = x[x]\star$, and $q_{23} = x[\star]x$. Then (u_1, q_1) and (u_2, q_{21}) are nested, as are (u_1, q_1) and (u_2, q_{23}) , while (u_1, q_1) and (u_2, q_{22}) are separated. Further, denoting $u_3 := [x]x$, then (u_3, q_3) with $q_3 = x\star$ is a placement of u_3 in w and the placements (u_1, q_1) and (u_3, q_3) are intersecting, while (u_2, q_{22}) and (u_3, q_3) are nested.

Theorem 3.5. [43, Theorem 4.11] *Let w be a bracketed word in $\mathfrak{M}(X)$. For any two placements (u_1, q_1) and (u_2, q_2) in w , exactly one of the following is true:*

- (a) (u_1, q_1) and (u_2, q_2) are separated;
- (b) (u_1, q_1) and (u_2, q_2) are nested;
- (c) (u_1, q_1) and (u_2, q_2) are intersecting.

Let a set Z be given. We next give the definition of monomial order on $\mathfrak{M}(Z)$.

Definition 3.6. A **monomial order** on $\mathfrak{M}(Z)$ is a well order $\leq := \leq_Z$ on $\mathfrak{M}(Z)$ such that

$$(26) \quad u < v \implies q|_u < q|_v, \quad \text{for all } u, v \in \mathfrak{M}(Z) \text{ and all } q \in \mathfrak{M}^*(Z).$$

Here, as usual, we denote $u < v$ if $u \leq v$ but $u \neq v$.

Since \leq is a well order, it follows from Eq. (26) that $1 \leq u$ and $u < [u]$ for all $u \in \mathfrak{M}(Z)$.

Definition 3.7. Let \leq be a monomial order on $\llbracket Z \rrbracket$, $f \in \mathbf{k} \llbracket Z \rrbracket$ and $S \subset \mathbf{k} \llbracket Z \rrbracket$. Let $\phi(x, y) := [x][y] - [B(x, y)] \in \mathbf{k} \llbracket x, y \rrbracket$.

- (a) The **leading bracketed word (monomial)** of f is the (unique) largest monomial \bar{f} appearing in f . The **leading coefficient** of f is the coefficient of \bar{f} in f , which we denote by $c(f)$. If $c(f) = 1$, we say f is **monic with respect to the monomial order \leq** . We define the **remainder** $R(f)$ of f by

$$(27) \quad R(f) := f - c(f)\bar{f}$$

so that $f = c(f)\bar{f} + R(f)$.

- (b) Suppose f is ϕ -reducible. We define the **leading ϕ -reducible monomial of f** to be the monomial $L(f)$ maximal with respect to \leq among monomials m appearing in f that are ϕ -reducible, that is,

$$L(f) := \max\{m \mid m \text{ is a monomial of } f \text{ and } m \notin \mathfrak{R}(Z)\}.$$

- (c) Suppose s is monic for all $s \in S$. We define the **rewriting system associated with S** to be the set of rewriting rules given by

$$(28) \quad \Pi_S(Z) := \{\bar{s} \rightarrow_S -R(s) \mid s \in S\}$$

and we denote the reflexive transitive closure of \rightarrow_S by $\xrightarrow{*}_S$. The set $\text{Irr}(S)$ of **irreducibles with respect to S and \leq** is defined by

$$\text{Irr}(S) := \text{Irr}^{Z, \leq}(S) = \mathfrak{M}(Z) \setminus \{q|_{\bar{s}} \mid q \in \mathfrak{M}^*(Z), s \in S\}.$$

An element $f \in \mathbf{k}\llbracket Z \rrbracket$ is **irreducible with respect to S** if $f \in \mathbf{k}\text{Irr}(S)$.

- (d) We say ϕ , or the rewriting system $\Pi_\phi(Z)$ defined by Eq. (14), is **compatible with \leq** if $\lfloor B(u, v) \rfloor < \lfloor u \rfloor \lfloor v \rfloor$ (equivalently, $\overline{\phi(u, v)} = \lfloor u \rfloor \lfloor v \rfloor$) for all $u, v \in \mathfrak{M}(Z)$.

Remark 3.8. When ϕ is compatible with \leq , and $S := S_\phi(Z)$ (as defined by Eq. (7)), the relation \rightarrow_ϕ (resp. its reflexive transitive closure $\xrightarrow{*}_\phi$) is the relation \rightarrow_S (resp. $\xrightarrow{*}_S$). If $s \in S$ is given by $s := \lfloor u \rfloor \lfloor v \rfloor - \lfloor B(u, v) \rfloor$, where $u, v \in \llbracket Z \rrbracket$, then the remainder $R(s)$ is the (q, u, v) -complement $R_{q, u, v}(s)$ of s . However, in general, if f has a monomial of the form $q|_{\lfloor u \rfloor \lfloor v \rfloor}$, the remainder $R(f)$ need not be the same as the (q, u, v) -complement $R_{q, u, v}(f)$ of f , unless $\bar{f} = q|_{\lfloor u \rfloor \lfloor v \rfloor}$. The set $\text{Irr}(S)$ is precisely the set of monomials that are in RBNF and $f \in \mathbf{k}\llbracket Z \rrbracket$ is ϕ -reducible if and only if f is S -reducible.

Lemma 3.9. Let Z be a set. Suppose $\phi := \lfloor x \rfloor \lfloor y \rfloor - \lfloor B(x, y) \rfloor$ is compatible with a monomial order \leq on $\mathfrak{M}(Z)$. Suppose $g, g' \in \mathbf{k}\llbracket Z \rrbracket$ are both ϕ -reducible and for $q \in \mathfrak{M}^*(Z)$ and $u, v \in \llbracket Z \rrbracket$, $g \xrightarrow{q, u, v}_\phi g'$. Then $L(g') \leq L(g)$, where equality holds if and only if $L(g) \neq q|_{\lfloor u \rfloor \lfloor v \rfloor}$.

Proof. We may write

$$g = (c_1 m_1 + \cdots + c_n m_n) \dot{+} h$$

for some integer $n \geq 1$; $c_1, \dots, c_n \in \mathbf{k}$ are all non-zero; $m_1 > \cdots > m_n$ are monomials in $\mathfrak{M}(Z) \setminus \mathfrak{R}(Z)$; and $h \in \mathbf{k}\llbracket Z \rrbracket$ is in RBNF. By Definition 3.7 (b), $L(g) = m_1$. Let i , ($1 \leq i \leq n$), be such that $m_i = q|_{\lfloor u \rfloor \lfloor v \rfloor}$. Then

$$\begin{aligned} g' &= c_i q|_{\lfloor B(u, v) \rfloor} \dot{+} R_{q, u, v}(g) \\ &= c_1 m_1 + \cdots + c_{i-1} m_{i-1} + c_i q|_{\lfloor B(u, v) \rfloor} + c_{i+1} m_{i+1} + \cdots + c_n m_n + h \end{aligned}$$

Now $m_1 \geq m_i = q|_{\lfloor u \rfloor \lfloor v \rfloor} > q|_{\lfloor B(u, v) \rfloor}$ since ϕ is compatible with \leq . Thus $L(g') = m_1 = L(g)$ if $i \neq 1$, and $L(g') < m_1$ if $i = 1$. \square

Theorem 3.10. Suppose $Z, B(x, y)$ and ϕ are as in Lemma 3.9. Then the rewriting system $\Pi_\phi(Z)$ is terminating.

Proof. Let

$$\mathcal{C} = \{g \in \mathbf{k}\llbracket Z \rrbracket \mid \text{there is an infinite } \phi\text{-reduction chain } g := g_0 \rightarrow_\phi g_1 \rightarrow_\phi \cdots\}.$$

We only need to prove that $\mathcal{C} = \emptyset$. Suppose that $\mathcal{C} \neq \emptyset$. Since g is ϕ -reducible for all $g \in \mathcal{C}$, and \leq is a well order on $\llbracket Z \rrbracket$, the set $\mathcal{L} := \{L(g) \mid g \in \mathcal{C}\}$, where $L(g)$ is the leading ϕ -reducible

monomial in g , is non-empty and has a least element w_0 . We fix a $g \in \mathcal{C}$ with $L(g) = w_0$ and fix an infinite ϕ -reduction chain $g := g_0 \xrightarrow{q_0, u_0, v_0} \phi g_1 \xrightarrow{q_1, u_1, v_1} \phi \cdots$. Then we have $g_i \in \mathcal{C}$ and hence ϕ -reducible for all $i \geq 1$. Let $w_i = L(g_i)$. By Lemma 3.9, we have $w_0 \geq w_1 \geq \dots$. Since every g_i is in \mathcal{C} , and w_0 is the least element in \mathcal{L} , we must have $w_0 = w_i$ for all i . By Lemma 3.9, none of the w_i is involved in ϕ -reduction of the fixed sequence starting with g . Let $f_i = g_i - b_i w_i$, where b_i is the coefficient of w_i in g_i . Then we have the infinite reduction sequence $f_0 \xrightarrow{q_0, u_0, v_0} \phi f_1 \xrightarrow{q_1, u_1, v_1} \phi \cdots$ and $L(f_0) < L(g)$. This is a contradiction, showing that $\mathcal{C} = \emptyset$. This completes the proof. \square

Now we apply Theorem 2.20 to our situation.

Lemma 3.11. *Let $\phi(x, y) := \lfloor x \rfloor \lfloor y \rfloor - \lfloor B(x, y) \rfloor \in \mathbf{k} \llbracket x, y \rrbracket$ with $B(x, y)$ in RBNF and totally linear in x, y . Let Z be a set and let $\Pi_\phi := \Pi_\phi(Z)$ be the rewriting system in Eq. (14). Let \leq be a monomial order on $\mathfrak{M}(Z)$ that is compatible with Π_ϕ . Let Y be a subset of $\mathfrak{M}(Z)$. Suppose that Π_ϕ restricts to a rewriting system $\Pi_{\phi, Y}$ on $\mathbf{k}Y \subseteq \mathbf{k} \llbracket Z \rrbracket$ in the sense of Definition 2.24, that is, for any $q \in \mathfrak{M}^*(Z)$ and $s \in S_\phi(Z)$, if $q|_{\bar{s}}$ is in Y , then $q|_{\bar{s}-s}$ is in $\mathbf{k}Y$. Suppose that $\Pi_{\phi, Y}$ is confluent. Let $q_i \in \mathfrak{M}^*(Z)$ and $s_i \in S_\phi(Z)$, $1 \leq i \leq n$, be such that $q_i|_{\bar{s}_i}$ is in Y and let $c_i \in \mathbf{k}$. If $c_1 q_1|_{\bar{s}_1-s_1} + c_2 q_2|_{\bar{s}_2-s_2} + \cdots + c_n q_n|_{\bar{s}_n-s_n} = 0$, then $c_1 q_1|_{\bar{s}_1} + c_2 q_2|_{\bar{s}_2} + \cdots + c_n q_n|_{\bar{s}_n} \xrightarrow{*}_{\Pi_\phi} 0$.*

Proof. Since Π_ϕ is compatible with the monomial order \leq , we have $\bar{s} = \lfloor u \rfloor \lfloor v \rfloor$ and $\bar{s}-s = \lfloor B(u, v) \rfloor$ for all $u, v \in \mathfrak{M}(Z)$. By Lemma 2.26, we have $q|_{\bar{s}} \dot{+} q|_{\bar{s}-s}$, and so the rewriting system $\Pi_{\phi, Y}$ is simple. In Theorem 2.20, take W to be Y . Then we obtain $c_1 q_1|_{\bar{s}_1} + c_2 q_2|_{\bar{s}_2} + \cdots + c_n q_n|_{\bar{s}_n} \xrightarrow{*}_{\Pi_{\phi, Y}} 0$. Thus the lemma follows. \square

Theorem 3.12. *Let \mathbf{k} be a field. Let $\phi(x, y) := \lfloor x \rfloor \lfloor y \rfloor - \lfloor B(x, y) \rfloor \in \mathbf{k} \llbracket x, y \rrbracket$ with $B(x, y)$ in RBNF and totally linear in x, y . Let Z be a set and let $\Pi_\phi := \Pi_\phi(Z)$ be the rewriting system in Eq. (14). Let \leq be a monomial order on $\mathfrak{M}(Z)$ that is compatible with $\Pi_\phi(Z)$. Then the following conditions are equivalent.*

- (a) *For all $u, v, w \in \llbracket Z \rrbracket$, the expression $B(B(u, v), w) - B(u, B(v, w))$ is ϕ -reducible to zero.*
- (b) *Π_ϕ is convergent.*

From the theorem we immediately obtain:

Corollary 3.13. *Let $\phi(x, y) := \lfloor x \rfloor \lfloor y \rfloor - \lfloor B(x, y) \rfloor \in \mathbf{k} \llbracket x, y \rrbracket$ with $B(x, y)$ in RBNF and totally linear in x, y . Let \leq be a monomial order on $\mathfrak{M}(Z)$ that is compatible with $\Pi_\phi(Z)$. Then $P = \lfloor \rfloor$ is a Rota-Baxter type operator if and only if $\Pi_\phi(Z)$ is convergent for every set Z .*

Proof. (\implies) If $P = \lfloor \rfloor$ is a Rota-Baxter type operator, then by Definition 2.31(d), the expression $B(B(u, v), w) - B(u, B(v, w))$ is ϕ -reducible to zero for all $u, v, w \in \llbracket Z \rrbracket$. Then by Theorem 3.12, Π_ϕ is convergent.

(\impliedby) If Π_ϕ is convergent, then by Theorem 3.12, the expression $B(B(u, v), w) - B(u, B(v, w))$ is ϕ -reducible to zero for all $u, v, w \in \llbracket Z \rrbracket$. Since the rewriting system $\Pi_\phi(Z)$ is compatible with the monomial order \leq , by Theorem 3.10, Π_ϕ is terminating. Together with the conditions that $B(x, y)$ is in RBNF and is totally linear in x, y , we see that $P = \lfloor \rfloor$ is a Rota-Baxter operator by definition. \square

We now give the proof of Theorem 3.12.

Proof. (a) \implies (b). Recall from Eqs. (6) and (7) that for $u, v \in \mathfrak{M}(Z)$,

$$\phi(u, v) := \lfloor u \rfloor \lfloor v \rfloor - \lfloor B(u, v) \rfloor, \quad \text{and} \quad S_\phi(Z) := \{ \phi(u, v) \mid u, v \in \mathfrak{M}(Z) \}.$$

Since the rewriting system $\Pi_\phi(Z)$ is compatible with the monomial order \leq , we have $\bar{s} = [u][v]$ for any $s \in S_\phi(Z)$. Then, for $f \in \mathfrak{M}(Z)$ and $g \in \mathbf{k}[\![Z]\!]$, $f \rightarrow_\phi g$ means that there are $q \in \mathfrak{M}^*(Z)$ and $s \in S_\phi(Z)$ such that $f = q|_{\bar{s}}$ and $g = q|_{\bar{s}-s}$.

By Theorem 3.10, the rewriting system Π_ϕ is terminating. By Lemma 2.17, to prove that Π_ϕ is confluent and hence convergent, we just need to prove that Π_ϕ is locally confluent. By Lemma 2.22, we only need to prove that Π_ϕ is locally term-confluent, i.e. for any local term fork $(ct \rightarrow_\phi cv_1, ct \rightarrow_\phi cv_2)$ where $(t, v_1), (t, v_2) \in \Pi_\phi$ and $c \in \mathbf{k}$, $c \neq 0$, we have $c(v_1 - v_2) \xrightarrow{*}_\phi 0$. Since \mathbf{k} is a field, a local term fork $(ct \rightarrow_\phi cv_1, ct \rightarrow_\phi cv_2)$ gives a local term fork $(t \rightarrow_\phi v_1, t \rightarrow_\phi v_2)$. Since $v_1 - v_2 \xrightarrow{*}_\phi 0$ implies $c(v_1 - v_2) \xrightarrow{*}_\phi 0$, we see that we only need to prove that,

for any local term fork $(t \rightarrow_\phi v_1, t \rightarrow_\phi v_2)$ where $(t, v_1), (t, v_2) \in \Pi_\phi$, we have $v_1 - v_2 \xrightarrow{*}_\phi 0$.

We will prove this statement by contradiction. Suppose there are local term forks $(t \rightarrow_\phi v_1, t \rightarrow_\phi v_2)$ such that $v_1 - v_2 \not\xrightarrow{*}_\phi 0$. Then the set

$$\mathfrak{N} := \left\{ f \in \mathfrak{M}(Z) \mid \text{there is a fork } f \rightarrow_\phi g_1, f \rightarrow_\phi g_2 \text{ with } g_1, g_2 \in \mathbf{k}[\![Z]\!] \text{ and } g_1 - g_2 \not\xrightarrow{*}_\phi 0 \right\}$$

is not empty. Since \leq is a well order on $\mathfrak{M}(Z)$, we can take f to be the least element in \mathfrak{N} with respect to \leq . Thus there are $q_1, q_2 \in \mathfrak{M}^*(Z)$ and $s_1, s_2 \in S_\phi(Z)$ such that

$$q_1|_{\bar{s}_1} = f = q_2|_{\bar{s}_2}, \quad g_1 = q_1|_{\bar{s}_1-s_1}, \quad g_2 = q_2|_{\bar{s}_2-s_2}$$

and $g_1 - g_2 \not\xrightarrow{*}_\phi 0$. Since $q_1|_{\bar{s}_1} = f = q_2|_{\bar{s}_2}$, \bar{s}_1 and \bar{s}_2 occur in f as bracketed subwords in the forms of placements (\bar{s}_1, q_1) and (\bar{s}_2, q_2) in f . By Theorem 3.5, these placements f have three possible relative locations. Accordingly, we will prove that $g_1 - g_2 \xrightarrow{*}_\phi 0$ in each of these three cases, yielding the desired contradiction. Before carrying out the proof, we fix some notations.

Let

$$Y := \{g \in \mathfrak{M}(Z) \mid g < f\}.$$

By the minimality of f in \mathfrak{N} , for $t \in Y$, all local term forks $(t \rightarrow_\phi v_1, t \rightarrow_\phi v_2)$ satisfy $v_1 - v_2 \xrightarrow{*}_\phi 0$. Note that since the monomial order \leq is compatible with Π_ϕ , v_1 and v_2 are in $\mathbf{k}Y$. In particular, Y is not empty. Then the rewriting system Π_ϕ , which is simple by Corollary 2.27, restricts to a rewriting system $\Pi_{\phi,Y}$ on $\mathbf{k}Y$ which is simple and locally term-confluent. By Lemma 2.22, $\Pi_{\phi,Y}$ on the space $\mathbf{k}Y$ is confluent.

Since s_1, s_2 are in $S_\phi(Z)$, using Eq. (16), there exist $u, v, r, t \in \mathfrak{M}(Z)$ such that

$$(29) \quad \begin{aligned} s_1 &= \phi(u, v) = [u][v] - [B(u, v)] = [u][v] - \sum_{i=1}^k a_i [B_i(u, v)], \\ s_2 &= \phi(r, t) = [r][t] - [B(r, t)] = [r][t] - \sum_{i=1}^k a_i [B_i(r, t)]. \end{aligned}$$

Case I. Suppose that the placements (\bar{s}_1, q_1) and (\bar{s}_2, q_2) are separated. Then by Definition 3.2, there is $q \in \mathfrak{M}^{\star 1, \star 2}(Z)$ such that

$$q|_{\bar{s}_1, \bar{s}_2} = f = q|_{\bar{s}_1} = q|_{\bar{s}_2}.$$

By $q|_{\star_1} = q|_{\star_1, \bar{s}_2}$ and $q|_{\star_2} = q|_{\bar{s}_1, \star_2}$, we have

$$g_1 = q|_{\bar{s}_1-s_1} = q|_{\bar{s}_1-s_1, \bar{s}_2} \quad \text{and} \quad g_2 = q|_{\bar{s}_2-s_2} = q|_{\bar{s}_1, \bar{s}_2-s_2}.$$

By Eqs. (29), we have

$$g_1 = \sum_{i=1}^k a_i q|_{[B_i(u,v)], \overline{s_2}} \quad \text{and} \quad g_2 = \sum_{i=1}^k a_i q|_{\overline{s_1}, [B_i(r,t)]},$$

and so

$$\begin{aligned} g_1 - g_2 &= \sum_{i=1}^k a_i q|_{[B_i(u,v)], \overline{s_2}} - \sum_{i=1}^k a_i q|_{\overline{s_1}, [B_i(r,t)]} \\ &= \sum_{\ell=1}^{2k} c_\ell q_\ell|_{\overline{u_\ell}}, \end{aligned}$$

where

$$\begin{aligned} q_\ell &:= \begin{cases} q|_{[B_\ell(u,v)], \star}, & 1 \leq \ell \leq k, \\ q|_{\star, [B_{\ell-k}(r,t)]}, & k+1 \leq \ell \leq 2k, \end{cases} & c_\ell &:= \begin{cases} a_\ell, & 1 \leq \ell \leq k, \\ -a_{\ell-k}, & k+1 \leq \ell \leq 2k. \end{cases} \\ u_\ell &:= \begin{cases} s_2, & 1 \leq \ell \leq k, \\ s_1, & k+1 \leq \ell \leq 2k, \end{cases} & \overline{u_\ell} &:= \begin{cases} \overline{s_2}, & 1 \leq \ell \leq k, \\ \overline{s_1}, & k+1 \leq \ell \leq 2k. \end{cases} \end{aligned}$$

Then we have

$$\begin{aligned} \sum_{\ell=1}^{2k} c_\ell q_\ell|_{\overline{u_\ell} - u_\ell} &= \sum_{i=1}^k a_i q|_{[B_i(u,v)], \overline{s_2} - s_2} - \sum_{i=1}^k a_i q|_{\overline{s_1} - s_1, [B_i(r,t)]} \\ &= \sum_{i,j=1}^k a_i q|_{[B_i(u,v)], [B_j(r,t)]} - \sum_{i,j=1}^k a_i q|_{[B_j(u,v)], [B_i(r,t)]} \\ &= 0. \end{aligned}$$

Since the monomial order \leq is compatible with Π_ϕ , we have

$$q_\ell|_{\overline{u_\ell}} < q|_{[B_\ell(u,v)], \overline{u_\ell}} < q|_{\overline{s_1}, \overline{s_2}} = f, \quad 1 \leq \ell \leq k$$

and

$$q_\ell|_{\overline{u_\ell}} < q|_{\overline{u_\ell}, [B_{\ell-k}(r,t)]} < q|_{\overline{s_1}, \overline{s_2}} = f, \quad k+1 \leq \ell \leq 2k.$$

Then $q_\ell|_{\overline{u_\ell}}$ are in Y for $1 \leq \ell \leq 2k$. By the compatibility of \leq with Π_ϕ again, $q_\ell|_{\overline{u_\ell}} > q_\ell|_{\overline{u_\ell} - u_\ell}$, and so $q_\ell|_{\overline{u_\ell} - u_\ell}$ are in $\mathbf{k}Y$ for $1 \leq \ell \leq 2k$. By Lemma 3.11, $g_1 - g_2 = \sum_{\ell=1}^{2k} c_\ell q_\ell|_{\overline{u_\ell}} \xrightarrow{*}_\phi 0$. This is a contradiction.

Case II. Suppose that the placements $(\overline{s_1}, q_1)$ and $(\overline{s_2}, q_2)$ are nested. Without loss of generality, assume that there exists $q \in \mathfrak{M}^*(Z)$ such that $q_1|_q = q_2$. We first consider the case when $q = \star$. Then $q_1 = q_2$. From $q_1|_{\overline{s_1}} = f = q_2|_{\overline{s_2}}$, we obtain $\overline{s_1} = \overline{s_2}$. Since $\overline{s_1} = [u][v]$ and $\overline{s_2} = [r][t]$, we must have, by Eq. (5), $[u] = [r]$ and $[v] = [t]$. So we have $u = r$, $v = t$, and $g_1 = q_1|_{[B(u,v)]} = q_2|_{[B(u,v)]} = g_2$, and there is nothing more to prove.

Now suppose that $q \neq \star$ and hence $q_1 \neq q_2$ and $\overline{s_1} \neq \overline{s_2}$. Since $\overline{s_1} = [u][v]$ and $\overline{s_2} = [r][t]$, by Eq. (25), there exists $q' \in \mathfrak{M}^*(Z)$ such that

- (i) either $u = q'|_{\overline{s_2}}$ and $q = [q']|v$;
- (ii) or $v = q'|_{\overline{s_2}}$ and $q = [u]|q'$.

In subcase (i), we have

$$g_1 - g_2 = q_1|_{\overline{s_1}-s_1} - q_2|_{\overline{s_2}-s_2} = q_1|_{\overline{s_1}-s_1} - (q_1|_q)|_{\overline{s_2}-s_2} = q_1|_{\overline{s_1}-s_1-q|_{\overline{s_2}-s_2}}.$$

Since $[0] = 0$, to show $g_1 - g_2 \xrightarrow{*}_\phi 0$, it suffices to prove that $\overline{s_1} - s_1 - q|_{\overline{s_2}-s_2} = q|_{s_2-s_1}$ (by Eq. (25)) is ϕ -reducible to zero. Applying the conditions given in Subcase (i) and expanding $B(x, y) = \sum_{i=1}^k a_i B_i(x, y)$ as a linear combination of distinct monomials $B_i(x, y)$ with non-zero coefficients $a_i \in \mathbf{k}$, we have

$$\begin{aligned} q|_{s_2-s_1} &= [q'|_{\overline{s_2}}][v] - [q'|_{[B(r,t)]}][v] - [u][v] + [B(u, v)] \\ &= -[q'|_{[B(r,t)]}][v] + [B(q'|_{[r][t]}, v)] \\ &= -\sum_{i=1}^k a_i [q'|_{[B_i(r,t)]}][v] + \sum_{i=1}^k a_i [B_i(q'|_{[r][t]}, v)] \\ &= \sum_{\ell=1}^{2k} c_\ell q_\ell|_{\overline{u_\ell}}, \end{aligned}$$

where

$$\begin{aligned} q_\ell &:= \begin{cases} \star, & 1 \leq \ell \leq k, \\ [B_{\ell-k}(q', v)], & k+1 \leq \ell \leq 2k, \end{cases} & c_\ell &:= \begin{cases} -a_\ell, & 1 \leq \ell \leq k, \\ a_{\ell-k}, & k+1 \leq \ell \leq 2k, \end{cases} \\ u_\ell &:= \begin{cases} \phi(q'|_{[B_\ell(r,t)]}, v), & 1 \leq \ell \leq k, \\ \phi(r, t), & k+1 \leq \ell \leq 2k, \end{cases} & \overline{u_\ell} &:= \begin{cases} [q'|_{[B_\ell(r,t)]}][v], & 1 \leq \ell \leq k, \\ [r][t], & k+1 \leq \ell \leq 2k. \end{cases} \end{aligned}$$

Then we have

$$\begin{aligned} \sum_{\ell=1}^{2k} c_\ell q_\ell|_{\overline{u_\ell}-u_\ell} &= \sum_{i=1}^k -a_i [B(q'|_{[B_i(r,t)]}, v)] + \sum_{i=1}^k a_i [B_i(q'|_{[B(r,t)]}, v)] \\ &= -\sum_{i,j=1}^k a_j a_i [B_j(q'|_{[B_i(r,t)]}, v)] + \sum_{i,j=1}^k a_i a_j [B_i(q'|_{[B_j(r,t)]}, v)] \\ &= 0. \end{aligned}$$

Note that $q_\ell|_{\overline{u_\ell}} < \overline{s_1} \leq q_1|_{\overline{s_1}} = f$ and hence is in Y . Applying Lemma 3.11 to $\mathbf{k}Y$ with u_ℓ , $1 \leq \ell \leq 2k$, given above, we obtain

$$q|_{s_2-s_1} = \sum_{\ell=1}^{2k} c_\ell q_\ell|_{\overline{u_\ell}} \xrightarrow{*}_\phi 0.$$

Therefore $g_1 - g_2 \xrightarrow{*}_\phi 0$. Subcase (ii) can be similarly treated. This is again a contradiction.

Case III. Suppose that the placements $(\overline{s_1}, q_1)$ and $(\overline{s_2}, q_2)$ are intersecting. Then by Definition 3.2, $q_1 \neq q_2$, and by Remark 3.3.(c), without loss of generality, we may assume that the partial overlap occurs at a right segment of $\overline{s_1}$ and a left segment of $\overline{s_2}$. Since $\overline{s_1} = [u][v]$ and $\overline{s_2} = [r][t]$, the common segment must be a proper subword, indeed, the subword $[v] = [r]$, and so $v = r$. We have $[u][v][t]$ appearing in f . Let $q \in \mathfrak{M}^{\star_1, \star_2}(Z)$ be the (\star_1, \star_2) -bracketed word obtained by replacing the occurrence of $[u][v]$ in f by \star_1 and the occurrence of $[t]$ in f by \star_2 (thus, \star_1 and \star_2 are adjacent symbols in q). More precisely, using the convention from Eqs. (21) and (22), we have

$$q_1 = q^{\star_2}|_{[t]} \quad \text{and} \quad q_2 = q^{\star_1}|_{[u]},$$

where in the first equation, we identify \star with \star_1 and in the second, \star with \star_2 . Let $p \in \llbracket Z \rrbracket^\star$ be the \star -bracketed word obtained by replacing $\star_1 \star_2$ in q by \star . Then using the convention from Eqs. (21) and (22), we have

$$\begin{aligned} g_1 - g_2 &= q_1|_{\overline{s_1-s_1}} - q_2|_{\overline{s_2-s_2}} \\ &= (q^{\star_2}|_{[t]})|_{(\overline{s_1-s_1})} - (q^{\star_1}|_{[u]})|_{(\overline{s_2-s_2})} \\ &= p|_{(\overline{s_1-s_1})[t]} - p|_{[u](\overline{s_2-s_2})} \\ &= p|_{((\overline{s_1-s_1})[t]-[u](\overline{s_2-s_2}))} \\ &= p|_{[B(u,v)][t]-[u][B(v,t)]} \\ &\xrightarrow{*}_\phi p|_{[B(B(u,v),t)]-[B(u,B(v,t))]}, \end{aligned}$$

where the last step follows from the rewriting

$$\begin{aligned} [B(u,v)][t] - [u][B(v,t)] &= [B(u,v)][t] \dot{+} (-[u][B(v,t)]) \\ &\rightarrow_\phi [B(B(u,v),t)] \dot{+} (-[u][B(v,t)]) \\ &\rightarrow_\phi [B(B(u,v),t)] - [B(u,B(v,t))]. \end{aligned}$$

By assumption (a), $B(B(u,v),t) - B(u,B(v,t)) \xrightarrow{*}_\phi 0$. So $g_1 - g_2 \xrightarrow{*}_\phi 0$. This is a contradiction.

This completes the proof of (a) \implies (b).

(b) \implies (a). Suppose that the rewriting system Π_ϕ is convergent. Then it is confluent. Thus for any $u, v, w \in \mathfrak{M}(Z)$, the fork

$$\begin{aligned} ([u][v][w] \rightarrow_\phi [u][B(v,w)] \rightarrow_\phi [B(u,B(v,w))]), \\ [u][v][w] \rightarrow_\phi [B(u,v)][w] \rightarrow_\phi [B(B(u,v),w)]) \end{aligned}$$

is joinable. Then by Theorem 2.20, we have $[B(B(u,v),w)] - [B(u,B(v,w))] \xrightarrow{*}_\phi 0$ and $B(B(u,v),w) - B(u,B(v,w)) \xrightarrow{*}_\phi 0$. This proves (a). \square

4. ROTA-BAXTER TYPE OPERATORS AND GRÖBNER-SHIRSHOV BASIS

We now characterize Rota-Baxter type operators in terms of Gröbner-Shirshov bases. The main theorem and its proof are given in Section 4.1. The application of the main theorem to the construction of free objects is provided in Section 4.2.

4.1. CD lemma and the main theorem. We provide some background and then state the main theorem on Gröbner-Shirshov bases for Rota-Baxter type operators.

Definition 4.1. Let \leq be a monomial order on $\llbracket Z \rrbracket$. Let $f, g \in \mathbf{k}\llbracket Z \rrbracket$ be distinct monic bracketed polynomials with respect to the monomial order \leq (See Definition 3.7). If there exist $\mu, \nu, w \in \mathfrak{M}(Z)$ such that $w = \overline{f}\mu = \nu\overline{g}$ with $\max\{|\overline{f}|, |\overline{g}|\} < |w| < |\overline{f}| + |\overline{g}|$, we call the operated polynomial

$$(f, g)_w^{\mu, \nu} := f\mu - \nu g$$

the **intersection composition of f and g with respect to (μ, ν)** . If there exist $q \in \mathfrak{M}^\star(Z)$ and $w \in \mathfrak{M}(Z)$ such that $w = \overline{f} = q|_{\overline{g}}$, we call the operated polynomial

$$(f, g)_w^q := f - q|_g$$

the **including composition of f and g with respect to q** .

Definition 4.2. Let \leq be a monomial order on $\mathbb{k}\llbracket Z \rrbracket$. Let S be a set of monic bracketed polynomials in $\mathbb{k}\llbracket Z \rrbracket$ and let $w \in \mathfrak{M}(Z)$ be a monomial. An operated polynomial $f \in \mathbb{k}\llbracket Z \rrbracket$ is called **trivial modulo S with bound w** or, in short, **trivial modulo (S, w)** if it can be expressed as $f = \sum_i c_i q_i|_{s_i}$ with $c_i \in \mathbf{k}$, $q_i \in \mathfrak{M}^*(Z)$, $s_i \in S$ (so $f \in \text{Id}(S)$ by Eq. (8)) and $q_i|_{\overline{s_i}} < w$.

Definition 4.3. A set $S \subseteq \mathbb{k}\llbracket Z \rrbracket$ of monic bracketed polynomials is called a **Gröbner-Shirshov basis with respect to \leq** if, for all pairs $f, g \in S$ with $f \neq g$, every intersection composition of the form $(f, g)_w^{\mu, \nu}$ is trivial modulo S with bound w , and every including composition of the form $(f, g)_w^q$ is trivial modulo S with bound w .

The following Composition-Diamond Lemma is the basic fact for our study of Rota-Baxter type operators.

Theorem 4.4. (Composition-Diamond Lemma)[13, 27] Let S be a set of monic bracketed polynomials in $\mathbb{k}\llbracket Z \rrbracket$. Then the following conditions are equivalent.

- (a) S is a Gröbner-Shirshov basis in $\mathbb{k}\llbracket Z \rrbracket$.
- (b) For every non-zero $f \in \text{Id}(S)$, $\overline{f} = q|_{\overline{s}}$ for some $q \in \mathfrak{M}^*(Z)$ and some $s \in S$.
- (c) For every non-zero $f \in \text{Id}(S)$, f can be expressed in **triangular form**, that is, in the form

$$(30) \quad f = c_1 q_1|_{s_1} + c_2 q_2|_{s_2} + \cdots + c_n q_n|_{s_n},$$

where for $1 \leq i \leq n$, $c_i \in \mathbf{k}$ ($c_i \neq 0$), $s_i \in S$, $q_i \in \mathfrak{M}^*(Z)$ and $q_1|_{\overline{s_1}} > q_2|_{\overline{s_2}} > \cdots > q_n|_{\overline{s_n}}$.

- (d) As \mathbf{k} -modules, $\mathbb{k}\llbracket Z \rrbracket = \mathbf{k} \cdot \text{Irr}(S) \oplus \text{Id}(S)$ and $\text{Irr}(S)$ is a \mathbf{k} -basis of $\mathbb{k}\llbracket Z \rrbracket / \text{Id}(S)$.

Example 4.5. Let $\phi \in \mathbb{k}\llbracket x, y \rrbracket$ be an OPI of Rota Baxter type and let $S = S_\phi(Z)$. Then by Proposition 2.7 $\mathbb{k}\llbracket Z \rrbracket / \text{Id}(S) = \mathbb{k}\llbracket Z \rrbracket / I_\phi(Z)$ is the free ϕ -algebra $\mathbf{k}_\phi\llbracket Z \rrbracket$. If $S_\phi(Z)$ is a Gröbner-Shirshov basis of $\mathbb{k}\llbracket Z \rrbracket$, then $\text{Irr}(S_\phi(Z))$ is a basis of $\mathbf{k}_\phi\llbracket Z \rrbracket$.

We next review more general reduction relations for operated polynomial algebras $\mathbb{k}\llbracket Z \rrbracket$ from [27]. These relations generalize those for polynomial algebras $\mathbf{k}[Z]$ ([4, Section 8.2]) and, under suitable conditions, we will show they include $\xrightarrow{*}_\phi$.

Definition 4.6. Let $s \in \mathbb{k}\llbracket Z \rrbracket$ be monic with leading term \overline{s} . We use s to define the following **reduction relation \rightarrow_s** : For $g, g' \in \mathbb{k}\llbracket Z \rrbracket$, let $g \rightarrow_s g'$ denote the relation that there are some $c \in \mathbf{k}$ ($c \neq 0$) and some $q \in \mathfrak{M}^*(Z)$ such that

- (a) $g = cq|_{\overline{s}} + R(g)$, where $R(g) := g - cq|_{\overline{s}} \in \mathbb{k}\llbracket Z \rrbracket$;
- (b) $g' = cq|_{\overline{s-s}} + R(g)$.

Equivalently, we say that $g \rightarrow_s g'$ if there are some $c \in \mathbf{k}$ ($c \neq 0$) and some $q \in \mathfrak{M}^*(Z)$ such that

- (a) $q|_{\overline{s}}$ is a monomial of g with coefficient c ;
- (b) $g' = g - cq|_{\overline{s}}$.

If S is a set of monic bracketed polynomials, we let $g \rightarrow_S g'$ denote the relation that $g \rightarrow_s g'$ for some $s \in S$. The reflexive transitive closure of \rightarrow_s and \rightarrow_S are denoted by $\xrightarrow{*}_s$ and $\xrightarrow{*}_S$, respectively.

Lemma 4.7. Let $\phi(x, y) := [x][y] - [B(x, y)] \in \mathbb{k}\llbracket x, y \rrbracket$ with $B(x, y)$ in RBNF and totally linear in x, y . Let Z be a set, let $S = S_\phi(Z)$ and let \leq be a monomial order on $\mathfrak{M}(Z)$ that is compatible with $\Pi_\phi(Z)$. Let $g \in \mathbb{k}\llbracket Z \rrbracket$ and $w \in \mathbb{k}\llbracket Z \rrbracket$ be given with $w > \overline{g}$. If $g \xrightarrow{*}_\phi 0$, then g is trivial modulo (S, w) .

Proof. By Theorem 3.10, the rewriting system Π_ϕ is terminating. From $g \xrightarrow{*}_\phi 0$, there exist $k \geq 1$ and $g_i \in \mathbf{k}\llbracket Z \rrbracket$, $1 \leq i \leq k$, such that

$$g =: g_1 \rightarrow_\phi g_2 \rightarrow_\phi \cdots \rightarrow_\phi g_k := 0.$$

Claim 4.8. We have $\overline{g_{i+1}} \leq \overline{g_i}$, $1 \leq i \leq k-1$.

Proof. For $1 \leq i \leq k$, we can rewrite g_i as

$$(31) \quad g_i = (c_{i,1}u_{i,1} + \cdots + c_{i,n_i}u_{i,n_i}) \dot{+} g_{i,2},$$

where for $1 \leq j \leq n_i$, $0 \neq c_{i,j} \in \mathbf{k}$, $u_{i,j}$ is not in RBNF, $u_{i,1} > \cdots > u_{i,n_i}$ with respect to \leq and $g_{i,2} \in \mathbf{k}\llbracket Z \rrbracket$ is in RBNF. For any reduction $g_i \rightarrow_\phi g_{i+1}$, $1 \leq i \leq k-1$, there exist $q_i \in \mathfrak{M}^*(Z)$, $u_i, v_i \in \mathfrak{M}(Z)$ and $0 \neq c_i \in \mathbf{k}$ such that

$$g_i = c_i q_i|_{[u_i][v_i]} \dot{+} R(g_i) \quad \text{and} \quad g_{i+1} = c_i q_i|_{[B(u_i, v_i)]} + R(g_i).$$

Then $q_i|_{[u_i][v_i]}$ is a monomial of g_i and is not in RBNF. Since $g_i = (c_{i,1}u_{i,1} + \cdots + c_{i,n_i}u_{i,n_i}) \dot{+} g_{i,2}$, there exists a natural number $1 \leq r_i \leq n_i$ such that $u_{i,r_i} = q_i|_{[u_i][v_i]}$ and $c_i = c_{i,r_i}$. Then

$$R(g_i) = (c_{i,1}u_{i,1} + \cdots + c_{i,n_i}u_{i,n_i} - c_{i,r_i}u_{i,r_i}) + g_{i,2}.$$

So we get

$$(32) \quad g_{i+1} = c_i q_i|_{[B(u_i, v_i)]} + (c_{i,1}u_{i,1} + \cdots + c_{i,n_i}u_{i,n_i} - c_{i,r_i}u_{i,r_i}) + g_{i,2}.$$

We distinguish two cases, depending on whether or not $\overline{g_i} = \overline{g_{i,2}}$.

Case 1. Suppose $\overline{g_i} = \overline{g_{i,2}}$: By Eq. (31), $\overline{g_{i,2}} > u_{i,1}$. By the compatibility of ϕ with \leq and the monomial property in Eq. (26), we have

$$u_{i,r_i} = q_i|_{[u_i][v_i]} > q_i|_{\overline{[B(u_i, v_i)]}}.$$

Since $u_{i,1} \geq u_{i,r_i}$, we have

$$\overline{g_{i,2}} > u_{i,1} \geq u_{i,r_i} > q_i|_{\overline{[B(u_i, v_i)]}}.$$

By $u_{i,1} > \cdots > u_{i,n_i}$ and Eq. (32), we get $\overline{g_{i+1}} = \overline{g_{i,2}}$. Thus $\overline{g_{i+1}} = \overline{g_i}$.

Case 2. Suppose $\overline{g_i} \neq \overline{g_{i,2}}$: Then $\overline{g_i} > \overline{g_{i,2}}$. By Eq. (31), we have $\overline{g_i} = u_{i,1}$ and then $\overline{g_{i,2}} < u_{i,1}$. By the compatibility of ϕ with \leq and the monomial property in Eq. (26), we have

$$u_{i,1} \geq u_{i,r_i} = q_i|_{[u_i][v_i]} > q_i|_{\overline{[B(u_i, v_i)]}}.$$

There are two subcases to consider. First assume that $u_{i,1} = u_{i,r_i}$. Then $u_{i,1} = q_i|_{[u_i][v_i]}$. By Eq. (32), we get

$$(33) \quad g_{i+1} = c_i q_i|_{[B(u_i, v_i)]} + (c_{i,2}u_{i,2} + \cdots + c_{i,n_i}u_{i,n_i}) + g_{i,2}.$$

So $\overline{g_{i+1}} \leq \max\{q_i|_{\overline{[B(u_i, v_i)]}}, u_{i,2}, \overline{g_{i,2}}\}$. Since $q_i|_{\overline{[B(u_i, v_i)]}} < u_{i,1}$, $u_{i,2} < u_{i,1}$ and $\overline{g_{i,2}} < u_{i,1}$, we have $\overline{g_{i+1}} < u_{i,1}$. Hence $\overline{g_{i+1}} < \overline{g_i}$. Next assume $u_{i,1} > u_{i,r_i}$. Then $2 \leq r_i \leq n_i$. By Eq. (32), we can rewrite g_{i+1} as

$$g_{i+1} = c_{i,1}u_{i,1} + c_{i,2}u_{i,2} + \cdots + c_{i,r_i}q_i|_{[B(u_i, v_i)]} + \cdots + c_{i,n_i}u_{i,n_i} + g_{i,2}.$$

Since $u_{i,1} > u_{i,r_i} > q_i|_{\overline{[B(u_i, v_i)]}}$, $u_{i,1} > \overline{g_{i,2}}$ and $u_{i,1} > u_{i,2} > \cdots > u_{i,n_i}$, we have $\overline{g_{i+1}} = u_{i,1}$. Thus $\overline{g_{i+1}} = \overline{g_i}$.

In summary, we have $\overline{g_{i+1}} \leq \overline{g_i}$, $1 \leq i \leq k-1$. □

Now we continue with the proof of Lemma 4.7. By $g_i \rightarrow_\phi g_{i+1}$ there exist $q_i \in \mathfrak{M}^*(Z)$, $s_i := \lfloor u_i \rfloor \lfloor v_i \rfloor - \lfloor B(u_i, v_i) \rfloor \in S$ and $0 \neq c_i$ such that $g_{i+1} = g_i - c_i q_i|_{s_i}$. So we have $g_i = g_{i+1} + c_i q_i|_{s_i}$. From the finite reduction sequence $g =: g_1 \rightarrow_\phi g_2 \rightarrow_\phi \cdots \rightarrow_\phi g_k := 0$, we get

$$\begin{aligned} g_1 &= g_2 + c_1 q_1|_{s_1} \\ &= g_3 + c_2 q_2|_{s_2} + c_1 q_1|_{s_1} \\ &\cdots \\ &= g_k + c_{k-1} q_{k-1}|_{s_{k-1}} + \cdots + c_1 q_1|_{s_1}. \end{aligned}$$

Since $g_k = 0$, we have

$$g_1 = c_1 q_1|_{s_1} + \cdots + c_{k-1} q_{k-1}|_{s_{k-1}}.$$

By Claim 4.8, we have

$$\overline{g_k} \leq \overline{g_{k-1}} \leq \cdots \leq \overline{g_1}.$$

Since $c_i q_i|_{s_i} = g_i - g_{i+1}$, we get $q_i|_{s_i} \leq \max\{\overline{g_i}, \overline{g_{i+1}}\} = \overline{g_i} \leq \overline{g_1} < w$ by our choice of w . This means that g is trivial modulo (S, w) . \square

Together with Corollary 3.13, the following theorem characterizes Rota-Baxter operators in terms of convergent rewriting systems and Gröbner-Shirshov bases.

Theorem 4.9. *Let $\phi(x, y) := \lfloor x \rfloor \lfloor y \rfloor - \lfloor B(x, y) \rfloor \in \mathbf{k}\llbracket x, y \rrbracket$ with $B(x, y)$ in RBNF and totally linear in x, y . Let Z be a set, and let \leq be a monomial order on $\mathfrak{M}(Z)$ that is compatible with $\Pi_\phi(Z)$. The following conditions are equivalent.*

- (a) *The rewriting system $\Pi_\phi(Z)$ is convergent.*
- (b) *With respect to \leq , the set $S := S_\phi(Z)$ is a Gröbner-Shirshov basis in $\mathbf{k}\llbracket Z \rrbracket$.*

Proof. (a) \implies (b) Suppose that the rewriting system Π_ϕ is convergent.

Let two elements f and g of $S_\phi(Z)$ be given with $f \neq g$. They are of the form

$$f := \phi(u, v), \quad g := \phi(r, s), \quad u, v, r, s \in \mathfrak{M}(Z).$$

Since Π_ϕ is compatible with \leq , we have $\overline{f} = \lfloor u \rfloor \lfloor v \rfloor$ and $\overline{g} = \lfloor r \rfloor \lfloor s \rfloor$.

(The case of intersection compositions). Suppose that $w = \overline{f}\mu = \nu\overline{g}$ gives an intersection composition, where $\mu, \nu \in \mathfrak{M}(Z)$. Since $|\overline{f}| = |\overline{g}| = 2$, we must have $|w| < |\overline{f}| + |\overline{g}| = 4$. Thus $|w| = 3$. This means that $|\mu| = |\nu| = 1$. Since f, g are monic, we have $\mu = \lfloor s \rfloor, \nu = \lfloor u \rfloor$. Thus $w = (\lfloor u \rfloor \lfloor v \rfloor) \lfloor s \rfloor = \lfloor u \rfloor (\lfloor r \rfloor \lfloor s \rfloor)$. Then we have $v = r$ and

$$\begin{aligned} (f, g)_w^{\mu, \nu} &= f\mu - \nu g \\ &= (\lfloor u \rfloor \lfloor v \rfloor - \lfloor B(u, v) \rfloor)\mu - \nu(\lfloor r \rfloor \lfloor s \rfloor - \lfloor B(r, s) \rfloor) \\ &= -(\lfloor B(u, v) \rfloor \lfloor s \rfloor - \lfloor u \rfloor \lfloor B(v, s) \rfloor). \end{aligned}$$

Since

$$\lfloor B(u, v) \rfloor \lfloor s \rfloor \not\leq \lfloor u \rfloor \lfloor B(v, s) \rfloor \rightarrow_\phi \lfloor u \rfloor \lfloor B(v, s) \rfloor$$

and Π_ϕ is convergent, by Theorem 2.20, we have

$$\lfloor B(u, v) \rfloor \lfloor s \rfloor - \lfloor u \rfloor \lfloor B(v, s) \rfloor \xrightarrow[\phi]{*} 0$$

Since

$$\overline{\lfloor B(u, v) \rfloor \lfloor s \rfloor - \lfloor u \rfloor \lfloor B(v, s) \rfloor} \leq \max\{\overline{\lfloor B(u, v) \rfloor \lfloor s \rfloor}, \overline{\lfloor u \rfloor \lfloor B(v, s) \rfloor}\} < \lfloor u \rfloor \lfloor v \rfloor \lfloor s \rfloor.$$

By Lemma 4.7, $\lfloor B(u, v) \rfloor \lfloor s \rfloor - \lfloor u \rfloor \lfloor B(v, s) \rfloor$ is trivial modulo $(S, \lfloor u \rfloor \lfloor v \rfloor \lfloor s \rfloor)$.

(The case of including compositions). Suppose that $w := \bar{f} = q|_{\bar{g}}$. Then we have $w := [u][v] = q|_{[r][s]}$. If $q = \star$, then $[u][v] = [r][s]$. Thus $u = r$ and $v = s$. Hence $f = \phi(u, v) = g$, a contradiction to the hypothesis that $f \neq g$. Then we get $q \neq \star$. So the \star in q can come from either u or v . Thus f and g could only have the following including compositions.

(a) If $u = q'|_{[r][s]}$ for some $q' \in \mathfrak{M}^\star(Z)$, then

$$\bar{f} = q|_{\bar{g}} = [q'|_{[r][s]}][v]$$

with $q := [q'] [v]$.

(b) If $v = q'|_{[r][s]}$ for some $q' \in \mathfrak{M}^\star(Z)$, then

$$\bar{f} = q|_{\bar{g}} = [u][q'|_{[r][s]}]$$

with $q := [u][q']$.

So we just need to check that in both cases these compositions are trivial modulo (S, w) . Consider the first case. Using the notation in Eq. (29), this composition is

$$\begin{aligned} (f, g)_w^q &= f - q|_g \\ &= [u][v] - \sum_{i=1}^k a_i [B_i(u, v)] - [q'|_g][v] \\ &= [q'|_{[r][s]}][v] - \sum_{i=1}^k a_i [B_i(q'|_{[r][s]}, v)] \\ &\quad - \left([q'|_{[r][s]}][v] - \sum_{i=1}^k a_i [q'|_{[B_i(r, s)]}[v] \right) \\ &= - \sum_{i=1}^k a_i [B_i(q'|_{[r][s]}, v)] + \sum_{i=1}^k a_i [q'|_{[B_i(r, s)]}[v] \\ &= - \sum_{i=1}^k a_i [B_i(q'|_{\phi(r, s)}, v)] - \sum_{i=1}^k a_i [B_i(q'|_{[B(r, s)]}, v)] \\ &\quad + \sum_{i=1}^k a_i \phi(q'|_{[B_i(r, s)]}, v) + \sum_{i=1}^k a_i [B(q'|_{[B_i(r, s)]}, v)] \\ &= - \sum_{i=1}^k a_i [B_i(q'|_{\phi(r, s)}, v)] + \sum_{i=1}^k a_i \phi(q'|_{[B_i(r, s)]}, v) \\ &\quad - \sum_{i=1}^k a_i \sum_{j=1}^k a_j [B_i(q'|_{[B_j(r, s)]}, v)] + \sum_{i=1}^k a_i \sum_{j=1}^k a_j [B_j(q'|_{[B_i(r, s)]}, v)] \\ &= - \sum_{i=1}^k a_i [B_i(q'|_{\phi(r, s)}, v)] + \sum_{i=1}^k a_i \phi(q'|_{[B_i(r, s)]}, v) \end{aligned}$$

since the double sums become the same after exchanging i and j . Let $q_i = \lfloor B_i(q', v) \rfloor \in \mathfrak{M}^*(Z)$, $1 \leq i \leq k$. Then

$$\sum_{i=1}^k a_i \lfloor B_i(q' |_{\phi(r,s)}, v) \rfloor = \sum_{i=1}^k a_i q_i |_{\phi(r,s)}.$$

Further

$$q_i |_{\overline{\phi(r,s)}} = q_i |_{\lfloor r \rfloor \lfloor s \rfloor} = \lfloor B_i(q' |_{\lfloor r \rfloor \lfloor s \rfloor}, v) \rfloor = \lfloor B_i(u, v) \rfloor < \lfloor u \rfloor \lfloor v \rfloor = w.$$

Thus the first sum is trivial modulo (S, w) .

For the second sum, we have

$$\sum_{i=1}^k a_i \phi(q' |_{\lfloor B_i(r,s) \rfloor}, v) = \sum_{i=1}^k a_i q_i |_{u_i},$$

where $q_i = \star$ and $u_i := \phi(q' |_{\lfloor B_i(r,s) \rfloor}, v) \in S$. Further,

$$q_i |_{\overline{u_i}} = \overline{u_i} = \overline{\phi(q' |_{\lfloor B_i(r,s) \rfloor}, v)} = \lfloor q' |_{\lfloor B_i(r,s) \rfloor} \rfloor \lfloor v \rfloor < \lfloor q' |_{\lfloor r \rfloor \lfloor s \rfloor} \rfloor \lfloor v \rfloor = w.$$

Hence the second sum is also trivial modulo (S, w) . This proves $(f, g)_w^q$ is trivial modulo (S, w) . The proof of the second case is similar.

(b) \implies (a) By Theorem 3.10, we conclude that Π_ϕ is terminating. By Corollary 3.13, it remains to verify that $B(B(u, v), w) - B(u, B(v, w)) \xrightarrow{*}_\phi 0$ for $u, v, w \in \mathfrak{M}(Z)$. We prove this by contradiction. Suppose that there are $u, v, w \in \mathfrak{M}(Z)$ such that $B(B(u, v), w) - B(u, B(v, w))$ is not ϕ -reducible to zero. Then we have $B(B(u, v), w) - B(u, B(v, w)) \rightarrow_\phi \cdots \rightarrow_\phi G$ where $G \neq 0$ is in RBNF. Thus $\lfloor B(B(u, v), w) \rfloor - \lfloor B(u, B(v, w)) \rfloor \rightarrow_\phi \cdots \rightarrow_\phi \lfloor G \rfloor$. Note that

$$\begin{aligned} & \lfloor B(B(u, v), w) \rfloor - \lfloor B(u, B(v, w)) \rfloor \\ = & \lfloor B(B(u, v), w) \rfloor - \lfloor B(u, v) \rfloor \lfloor w \rfloor + \lfloor B(u, v) \rfloor \lfloor w \rfloor - \lfloor u \rfloor \lfloor v \rfloor \lfloor w \rfloor \\ & - \lfloor B(u, B(v, w)) \rfloor + \lfloor u \rfloor \lfloor B(v, w) \rfloor - \lfloor u \rfloor \lfloor B(v, w) \rfloor + \lfloor u \rfloor \lfloor v \rfloor \lfloor w \rfloor \end{aligned}$$

is in $\text{id}(S)$. Hence $\lfloor G \rfloor$ is in $\text{Id}(S)$. Since S is a Gröbner-Shirshov basis, by Theorem 4.4, there are $q \in \mathfrak{M}(Z)$ and $s \in S$ such that $\overline{\lfloor G \rfloor} = q |_{\overline{s}}$. This shows that $\lfloor G \rfloor$ is not in RBNF. Hence G is not in RBNF, a contradiction. In summary, ϕ is of Rota-Baxter type. Thus by Corollary 3.13, (a) follows.

This completes the proof of Theorem 4.9. \square

4.2. Construction of free ϕ -algebra. We next give the following explicit construction of free objects in the category of algebras with a given Rota-Baxter type operator. As we will see in Theorem 5.10, this construction applies to all the operators in the list of Conjecture 2.37 and thus provides a uniform exposition compared with the previously separate case-by-case construction method [1, 18, 24, 31, 41, 42].

Recall from Proposition 2.7 that $\mathbf{k}_\phi \llbracket Z \rrbracket = \mathbf{k} \llbracket Z \rrbracket / I_\phi(Z)$ is the free ϕ -algebra on Z . Let $\mathfrak{R}(Z)$ be the set of bracketed words in $\mathfrak{M}(Z)$ in RBNF. Then $\mathfrak{R}(Z)$ is closed under the operator $P_r := \lfloor \cdot \rfloor$. Let $\mathbf{k} \mathfrak{R}(Z)$ be the free \mathbf{k} -module with basis $\mathfrak{R}(Z)$ and let the operator P_r on $\mathfrak{R}(Z)$ be extended \mathbf{k} -linearly to $\mathbf{k} \mathfrak{R}(Z)$. Then $(\mathbf{k} \mathfrak{R}(Z), P_r)$ is an operated \mathbf{k} -module as defined in Definition 2.1.

Theorem 4.10. *Let $\phi(x, y) := \lfloor x \rfloor \lfloor y \rfloor - \lfloor B(x, y) \rfloor \in \mathbf{k} \llbracket x, y \rrbracket$ be of Rota-Baxter type. Let Z be a set and let \leq be a monomial order on $\mathfrak{M}(Z)$. Suppose that ϕ is compatible with \leq . Then:*

(a) *The composition of natural \mathbf{k} -module morphisms*

$$\mathbf{k}\mathfrak{R}(Z) \xrightarrow{\iota} \mathbf{k}\mathfrak{M}(Z) \equiv \mathbf{k}\llbracket Z \rrbracket \xrightarrow{\eta} \mathbf{k}_\phi\llbracket Z \rrbracket$$

is an isomorphism and $\eta \circ \iota(\mathfrak{R}(Z))$ is a \mathbf{k} -basis of $\mathbf{k}_\phi\llbracket Z \rrbracket$.

(b) Let $\alpha : \mathbf{k}_\phi\llbracket Z \rrbracket \rightarrow \mathbf{k}\mathfrak{R}(Z)$ be the inverse of $\eta \circ \iota$ from Part (a) and let Red be the composition $\mathbf{k}\llbracket Z \rrbracket \xrightarrow{\eta} \mathbf{k}_\phi\llbracket Z \rrbracket \xrightarrow{\alpha} \mathbf{k}\mathfrak{R}(Z)$. Then $(\mathbf{k}\mathfrak{R}(Z), \diamond, P_r)$ is a free ϕ -algebra, where the multiplication \diamond on $\mathbf{k}\mathfrak{R}(Z)$ is defined on $\mathfrak{R}(Z)$ as follows and extended by bilinearity. For any bracketed words $u, v \in \mathfrak{R}(Z)$:

- (i) $1 \diamond u = u \diamond 1 := u$, where 1 is the empty word in $\mathfrak{M}(Z)$;
- (ii) $u \diamond v := uv$ if either $u \in S(Z)$ or $v \in S(Z)$;
- (iii) $u \diamond v := [\text{Red}(B(u^*, v^*))]$ if $u = [u^*]$ and $v = [v^*]$ are both in $[\mathfrak{R}(Z)]$;
- (iv) $u \diamond v := u_1 \cdots u_{s-1}(u_s \diamond v_1)v_2 \cdots v_t$ if the standard decomposition $u_1 \cdots u_{s-1}u_s$ of u has $s > 1$ or the standard decomposition $v_1v_2 \cdots v_t$ of v has $t > 1$. Here except for $u_s \diamond v_1$, the rest are concatenations as in the standard decompositions defined in Eq. (15).

Proof. (a) By Corollary 3.13 and Theorem 4.9, $S = S_\phi(Z)$ is a Gröbner-Shirshov basis in $\mathbf{k}\llbracket Z \rrbracket$ with respect to \leq . Hence by Theorem 4.4, $\text{Irr}(S)$ is a \mathbf{k} -basis of $\mathbf{k}\llbracket Z \rrbracket / \text{Id}(S)$. Since $\text{Irr}(S) = \mathfrak{R}(Z)$ and $\text{Id}(S) = I_\phi(Z)$ from their definitions, Part (a) follows.

Before proving Theorem 4.10 (b), we give the following lemma.

Lemma 4.11. *We keep the same notations as in Theorem 4.10 (b).*

- (a) *The linear maps ι, η, α and Red are operated \mathbf{k} -module morphisms.*
- (b) *For any $f, g \in \mathbf{k}\llbracket Z \rrbracket$ such that $f \rightarrow_\phi g$, we have $\text{Red}(f) = \text{Red}(g)$;*
- (c) *For any $f \in \mathbf{k}\llbracket Z \rrbracket$, there exists $f' \in \mathbf{k}\llbracket Z \rrbracket$ in RBNF such that $f \xrightarrow{*}_\phi f'$, and $\text{Red}(f) = f'$.*

Proof. (a) By definition, $\mathfrak{R}(Z)$ is closed under taking the bracket. So $\mathbf{k}\mathfrak{R}(Z)$ is an operated \mathbf{k} -module. Then the embedding $\iota : \mathbf{k}\mathfrak{R}(Z) \rightarrow \mathbf{k}\llbracket Z \rrbracket$ is an operated \mathbf{k} -module morphism. Since $\eta : \mathbf{k}\llbracket Z \rrbracket \rightarrow \mathbf{k}_\phi\llbracket Z \rrbracket$ is a quotient map of operated \mathbf{k} -modules and hence an operated \mathbf{k} -module morphism, the composition $\eta \circ \iota$ is also one. Since $\eta \circ \iota$ is a linear bijection by Theorem 4.10(a), its inverse α is also an operated \mathbf{k} -module morphism. Then the composition $\text{Red} := \alpha \circ \eta$ is also one.

(b) By $f \rightarrow_\phi g$ and Definition 2.23, there exist $q \in \mathfrak{M}^*(Z)$, $u, v \in \mathfrak{R}(Z)$ and $0 \neq c \in \mathbf{k}$ such that

$$g = f - cq|_{([\![u]\!]v] - [B(u,v)])}.$$

Then $f - g = cq|_{([\![u]\!]v] - [B(u,v)])}$ is in $\text{Id}(S)$. So $f - g + \text{Id}(S) = 0 + \text{Id}(S)$ is in $\mathbf{k}_\phi\llbracket Z \rrbracket$. Then by Lemma 4.11.(a), we have

$$\text{Red}(f) - \text{Red}(g) = \text{Red}(f - g) = \alpha(\eta(f - g)) = \alpha(f - g + \text{Id}(S)) = 0.$$

(c) By Corollary 3.13, Π_ϕ is convergent. Hence Π_ϕ is terminating. Then there exists $f' \in \mathbf{k}\llbracket Z \rrbracket$ in RBNF such that $f \xrightarrow{*}_\phi f'$. We can assume that the reduction relation $f \xrightarrow{*}_\phi f'$ is given by the finite reduction sequence $f \rightarrow_\phi f_1 \rightarrow_\phi f_2 \rightarrow_\phi \cdots \rightarrow_\phi f'$. By Lemma 4.11 (b), we have

$$\text{Red}(f) = \text{Red}(f_1) = \text{Red}(f_2) = \cdots = \text{Red}(f').$$

Since f' is in RBNF, we have $\eta(f') = (\eta \circ \iota)(f') = \alpha^{-1}(f')$. So

$$\text{Red}(f') = (\alpha \circ \eta)(f') = \alpha(\alpha^{-1}(f')) = f'.$$

□

(b) We now prove Theorem 4.10(b). Since $\alpha : \mathbf{k}_\phi\llbracket Z \rrbracket \rightarrow \mathbf{k} \cdot \mathfrak{R}(Z)$ is an operated \mathbf{k} -module isomorphism by Theorem 4.10(a) and Lemma 4.11(a), we can transport the structure of a free ϕ -algebra on $\mathbf{k}_\phi\llbracket Z \rrbracket$ to $\mathbf{k} \cdot \mathfrak{R}(Z)$. More precisely, denote the multiplication and the linear operator on the free ϕ -algebra $\mathbf{k}_\phi\llbracket Z \rrbracket$ by \odot and $P_Z := \lfloor \rfloor \pmod{\text{Id}(S)}$ respectively. We define

$$u \diamond' v := \alpha(\alpha^{-1}(u) \odot \alpha^{-1}(v)) \quad \text{and} \quad P'_r(u) := \alpha(P_Z(\alpha^{-1}(u))) \quad \text{for any } u, v \in \mathfrak{R}(Z).$$

Then $(\mathbf{k} \cdot \mathfrak{R}(Z), \diamond', P'_r)$ is a ϕ -algebra isomorphic to $(\mathbf{k}_\phi\llbracket Z \rrbracket, \odot, P_Z)$, and hence is a free ϕ -algebra on Z .

Since $u \in \mathfrak{R}(Z)$ and $\alpha^{-1}(u) = u + \text{Id}(S)$, we have

$$P'_r(u) = \alpha(P_Z(\alpha^{-1}(u))) = \alpha(\lfloor u \rfloor + \text{Id}(S)) = \lfloor u \rfloor = P_r(u).$$

Thus, we get $P'_r = P_r = \lfloor \rfloor$.

It remains to prove that $u \diamond' v = u \diamond v$ for any $u, v \in \mathfrak{R}(Z)$. Note that $\alpha^{-1}(u) = u + \text{Id}(S) = \eta(u)$ and $\alpha^{-1}(v) = v + \text{Id}(S) = \eta(v)$. By Lemma 4.11(a), $\eta : \mathbf{k}\llbracket Z \rrbracket \rightarrow \mathbf{k}_\phi\llbracket Z \rrbracket$ is an operated \mathbf{k} -algebra homomorphism. Hence we have

$$\alpha^{-1}(u) \odot \alpha^{-1}(v) = \eta(u) \odot \eta(v) = \eta(uv).$$

Thus

$$u \diamond' v = \alpha(\alpha^{-1}(u) \odot \alpha^{-1}(v)) = \alpha(\eta(uv)) = \text{Red}(uv).$$

So we just need to show that $\text{Red}(uv) = u \diamond v$. For any given $u, v \in \mathfrak{R}(Z)$, let $u = u_1 \cdots u_{s-1} u_s$ and $v = v_1 v_2 \cdots v_t$ be the standard decompositions defined in Eq. (15). Then u_i is alternately in $S(Z)$ or in $\lfloor \mathfrak{R}(Z) \rfloor$, $1 \leq i \leq s$, and v_j is also alternately in $S(Z)$ or in $\lfloor \mathfrak{R}(Z) \rfloor$, $1 \leq j \leq t$. First consider $s = t = 1$. If $v = 1$ is the empty word in $\mathfrak{R}(Z)$, then

$$\text{Red}(u1) = \text{Red}(u) = u.$$

Thus, we have $\text{Red}(u1) = u = u \diamond 1$. Similarly, $\text{Red}(1u) = u = 1 \diamond u$. If $u \in S(Z)$ or $v \in S(Z)$, then $uv \in \mathfrak{R}(Z)$. Thus we have $\text{Red}(uv) = uv = u \diamond v$. If $u = \lfloor u^* \rfloor$ and $v = \lfloor v^* \rfloor$ with $u^*, v^* \in \mathfrak{R}(Z)$, then we have $uv = \lfloor u^* \rfloor \lfloor v^* \rfloor \rightarrow_\phi \lfloor B(u^*, v^*) \rfloor$. By Lemma 4.11(b), we get $\text{Red}(uv) = \text{Red}(\lfloor B(u^*, v^*) \rfloor)$. By Lemma 4.11(a), Red preserves the brackets. Then we get

$$(34) \quad \text{Red}(\lfloor u^* \rfloor \lfloor v^* \rfloor) = \text{Red}(\lfloor B(u^*, v^*) \rfloor) = \lfloor \text{Red}(B(u^*, v^*)) \rfloor =: u \diamond v.$$

Now we consider $s > 1$ or $t > 1$. If $u_s \in S(Z)$ or $v_1 \in S(Z)$, then $u_s v_1$ is in $\mathfrak{R}(Z)$. Thus, $uv = u_1 \cdots u_{s-1} u_s v_1 v_2 \cdots v_t$ is in $\mathfrak{R}(Z)$. Then we have

$$\text{Red}(uv) = uv = u_1 \cdots u_{s-1} (u_s \diamond v_1) v_2 \cdots v_t = u \diamond v.$$

If $u_s = \lfloor u_s^* \rfloor$ and $v_1 = \lfloor v_1^* \rfloor$ with $u_s^*, v_1^* \in \mathfrak{R}(Z)$, then u_{s-1} and v_2 , if exist, are in $S(Z)$. Since $uv = u_1 \cdots u_{s-1} (\lfloor u_s^* \rfloor \lfloor v_1^* \rfloor) v_2 \cdots v_t$ while $u_1 \cdots u_{s-1}$ and $v_2 \cdots v_t$ are in $\mathfrak{R}(Z)$, the rewriting system Π_ϕ can only be applied to $\lfloor u_s^* \rfloor \lfloor v_1^* \rfloor$. Since Π_ϕ is terminating, there exists $h \in \mathbf{k}\llbracket Z \rrbracket$ in RBNF such that

$$(35) \quad \lfloor u_s^* \rfloor \lfloor v_1^* \rfloor \rightarrow_\phi \lfloor B(u_s^*, v_1^*) \rfloor \xrightarrow{*}_\phi \lfloor h \rfloor.$$

Then we have

$$\begin{aligned} uv &= u_1 \cdots u_{s-1} (\lfloor u_s^* \rfloor \lfloor v_1^* \rfloor) v_2 \cdots v_t \\ &\rightarrow_\phi u_1 \cdots u_{s-1} (\lfloor B(u_s^*, v_1^*) \rfloor) v_2 \cdots v_t \\ &\xrightarrow{*}_\phi u_1 \cdots u_{s-1} \lfloor h \rfloor v_2 \cdots v_t. \end{aligned}$$

Since $[h]$ is in RBNF and u_{s-1} and v_2 are in $S(Z)$, $u_1 \cdots u_{s-1}[h]v_2 \cdots v_t$ is in $\mathbf{k}\mathfrak{R}(Z)$. Then by Lemma 4.11(c), we have

$$\text{Red}(uv) = u_1 \cdots u_{s-1}[h]v_2 \cdots v_t.$$

By Eq.(35) and Lemma 4.11.(b), we have

$$\text{Red}([u_s^*][v_1^*]) = \text{Red}([B(u_s^*, v_1^*)]) = [h].$$

Since $\text{Red} = \alpha \circ \eta$ is an operated \mathbf{k} -module morphism, $\text{Red}([B(u_s^*, v_1^*)]) = [\text{Red}(B(u_s^*, v_1^*))]$. So $[h] = [\text{Red}(B(u_s^*, v_1^*))]$. Thus, we get

$$\begin{aligned} \text{Red}(uv) &= \text{Red}(u_1 \cdots u_{s-1}[u_s^*][v_1^*]v_2 \cdots v_t) \\ &= u_1 \cdots u_{s-1}[h]v_2 \cdots v_t \\ &= u_1 \cdots u_{s-1}[\text{Red}(B(u_s^*, v_1^*))]v_2 \cdots v_t \\ &= u_1 \cdots u_{s-1}(u_s \diamond v_1)v_2 \cdots v_t \quad (\text{by Eq. (34)}) \\ &=: u \diamond v. \end{aligned}$$

Hence, $u \diamond' v = \text{Red}(uv) = u \diamond v$ for any $u, v \in \mathfrak{R}(Z)$. Then $(\mathbf{k}\mathfrak{R}(Z), \diamond, P_r) = (\mathbf{k}\mathfrak{R}(Z), \diamond', P'_r)$ and hence is a free ϕ -algebra. \square

5. APPLICATIONS TO CONJECTURE 2.37

We next construct a monomial order on $\mathfrak{M}(Z)$ that is compatible with the linear operators in Conjecture 2.37. This allows us to show that these operators are indeed Rota-Baxter type operators as claimed by the conjecture. At the same time this gives, in one stroke, an explicit construction of free objects in the categories of algebras with any of these operators. In the case of the Rota-Baxter operator, Nijenhuis operator or TD operator, such a construction was obtained previously by different methods [18, 24, 31, 41, 42]. See [13] for the construction of free Rota-Baxter algebras by the method of Gröbner-Shirshov basis.

5.1. Monomial order on $\mathfrak{M}(Z)$. We now construct a monomial order on $\mathfrak{M}(Z)$.

Let Z be a set with a well order \leq_Z . For $u = u_1 \cdots u_r \in M(Z)$ with $u_1, \dots, u_r \in Z$, define $\deg_Z(u) = r$. Note that $\deg_Z(1) = 0$. Define the **degree lexicographical order** \leq_{dlex} on $M(Z)$ by taking, for any $u = u_1 \cdots u_r, v = v_1 \cdots v_s \in M(Z) \setminus \{1\}$, where $u_1, \dots, u_r, v_1, \dots, v_s \in Z$,

$$u \leq_{\text{dlex}} v \Leftrightarrow \begin{cases} \deg_Z(u) < \deg_Z(v), \\ \text{or } \deg_Z(u) = \deg_Z(v) (= r) \text{ and } u_1 \cdots u_r \leq_{\text{lex}} v_1 \cdots v_r, \end{cases}$$

where \leq_{lex} is the lexicographical order on $M(Z)$, with the convention that the empty word $1 \leq_{\text{dlex}} u$ for all $u \in M(Z)$. Then we have

Lemma 5.1. [4] *If \leq_Z is a well order on Z , then \leq_{dlex} is a well order on $M(Z)$.*

Definition 5.2. Let Y be a nonempty set.

- (a) A **preorder** or **quasiorder** \leq_Y on Y is a binary relation that is reflexive and transitive, that is, for all $x, y, z \in Y$, we have
 - (i) $x \leq_Y x$; and
 - (ii) if $x \leq_Y y, y \leq_Y z$, then $x \leq_Y z$.
 We denote $x =_Y y$ if $x \leq_Y y$ and $x \geq_Y y$. If $x \leq_Y y$ but $x \neq_Y y$, we write $x <_Y y$ or $y >_Y x$.
- (b) A **pre-linear order** \leq_Y on Y is a preorder \leq_Y such that either $x \leq_Y y$ or $x \geq_Y y$ for all $x, y \in Y$.

We define the composition of two or more preorders.

Definition 5.3. (a) Let $k \geq 1$ and let $\leq_{\alpha_i}, 1 \leq i \leq k$, be preorders on a set Y . Let $u, v \in Y$. Recursively define

$$(36) \quad u \leq_{\alpha_1, \dots, \alpha_k} v \Leftrightarrow \begin{cases} u <_{\alpha_1} v, \\ \text{or } u =_{\alpha_1} v \text{ and } u \leq_{\beta} v, \end{cases}$$

where $\leq_{\beta} := \leq_{\alpha_2, \dots, \alpha_k}$ is defined by the induction hypothesis, with the convention that \leq_{β} is the trivial relation when $k = 1$, namely $u \leq_{\beta} v$ for all $u, v \in Y$.

(b) Let $k \geq 1$ and let $(Y_i, \leq_{Y_i}), 1 \leq i \leq k$, be partially ordered sets. Define the **lexicographical product order** \leq_{cllex} on the cartesian product $Y_1 \times Y_2 \times \dots \times Y_k$ by recursively defining

$$(37) \quad (x_1, x_2, \dots, x_k) \leq_{\text{cllex}} (y_1, y_2, \dots, y_k) \Leftrightarrow \begin{cases} x_1 <_{Y_1} y_1, \\ \text{or } x_1 = y_1 \text{ and } (x_2, \dots, x_k) \leq_{\text{cllex}} (y_2, \dots, y_k), \end{cases}$$

where $(x_2, \dots, x_k) \leq_{\text{cllex}} (y_2, \dots, y_k)$ is defined by the induction hypothesis, with the convention that \leq_{cllex} is the trivial relation when $k = 1$.

(c) Let $u = u_0[u_1^*]u_1[u_2^*] \dots [u_r^*]u_r, v = v_0[v_1^*]v_1[v_2^*] \dots [v_s^*]v_s \in \mathfrak{M}(Z)$, where $u_0, u_1, \dots, u_r, v_0, v_1, \dots, v_s \in M(Z)$ and $u_1^*, u_2^*, \dots, u_r^*, v_1^*, v_2^*, \dots, v_s^* \in \mathfrak{M}(Z)$. Define

$$(38) \quad u \leq_{\text{dgp}} v \Leftrightarrow \deg_P(u) \leq \deg_P(v),$$

where the **P -degree** $\deg_P(u)$ of u is the number of occurrence of $P = \lfloor \rfloor$ in u . Define

$$(39) \quad u \leq_{\text{brp}} v \Leftrightarrow r \leq s \quad (\text{that is } |u|_P \leq |v|_P),$$

where $|u|_P$ is the P -breadth defined after Eq. (5).

Lemma 5.4. (a) Let $k \geq 1$. Let $\leq_{\alpha_1}, \dots, \leq_{\alpha_{k-1}}$ be pre-linear orders on Z with descending chain condition (i.e., each decreasing chain in (Z, \leq_{α_i}) stabilizes after finitely many steps), and \leq_{α_k} is a well order on Z . Then the order $\leq_{\alpha_1, \dots, \alpha_k}$ is a well order on Z .

(b) [29] Let \leq_{Y_i} be a well order on $Y_i, 1 \leq i \leq k, k \geq 1$. Then the lexicographical product order \leq_{cllex} is a well order on the cartesian product $Y_1 \times Y_2 \times \dots \times Y_k$.

(c) The pre-linear orders \leq_{dgp} and \leq_{brp} satisfy the descending chain condition on $\mathfrak{M}(Z)$.

Proof. (a) We prove the claim by induction on $k \geq 1$. When $k = 1$, $\leq_{\alpha_1, \dots, \alpha_k}$ is a well order by the assumption. Assume that the claim holds for $k \leq n$ where $n \geq 1$ and consider the case when $k = n + 1$. Denote $\leq_{\beta} := \leq_{\alpha_2, \dots, \alpha_{n+1}}$. Then \leq_{β} is a well order by the induction hypothesis. We first show that $\leq_{\alpha_1, \beta}$ is a linear order. For all $u, v \in Z$, we have $u \leq_{\alpha_1} v$ or $u \geq_{\alpha_1} v$ since \leq_{α_1} is a pre-linear order. If $u \neq_{\alpha_1} v$, then we have $u <_{\alpha_1} v$ or $u >_{\alpha_1} v$. Thus we obtain $u <_{\alpha_1, \beta} v$ or $u >_{\alpha_1, \beta} v$ and we are done. If $u =_{\alpha_1} v$, then $u \geq_{\beta} v$ or $u \leq_{\beta} v$ since \leq_{β} is a linear order. Thus we have $u \geq_{\alpha_1, \beta} v$ or $u \leq_{\alpha_1, \beta} v$. Therefore, $\leq_{\alpha_1, \beta}$ is a linear order.

Thus we just need to prove that the order $\leq_{\alpha_1, \dots, \alpha_k}$ satisfies the descending chain condition. Suppose that $u_1 \geq_{\alpha_1, \beta} u_2 \geq_{\alpha_1, \beta} \dots$. Since \leq_{α_1} has descending chain condition, there exists $t \geq 1$ such that $u_t =_{\alpha_1} u_{t+1} =_{\alpha_1} \dots$. Thus we must have $u_t \geq_{\beta} u_{t+1} \geq_{\beta} \dots$. By the induction hypothesis, \geq_{β} is a well order and hence satisfies the descending chain condition. Thus the descending chain $u_1 \geq_{\alpha_1, \beta} u_2 \geq_{\alpha_1, \beta} \dots$ stabilizes after finite steps. Therefore, $\leq_{\alpha_1, \beta}$ is a well order. This completes the induction.

(b) is proved in [29, Chapter 4, Theorem 1.13].

(c) follows since both $\deg_P(u)$ and $|u|_P$ (defined after Eq. (5)) take values in $\mathbb{Z}_{\geq 0}$ and hence satisfy the descending chain condition. \square

For $m \geq 0$, denote

$$\mathfrak{M}^m(Z) = \{u \in \mathfrak{M}(Z) \mid |u|_P = m\},$$

Also denote $\mathfrak{M}_n^m(Z) = \mathfrak{M}_n(Z) \cap \mathfrak{M}^m(Z)$, where $n \geq 0$. For $u, v \in \mathfrak{M}_n^m(Z)$, define

$$(40) \quad u \leq_{\text{lex}_n} v \Leftrightarrow (u_1^*, u_2^*, \dots, u_m^*, u_0, \dots, u_m) \leq_{\text{clex}} (v_1^*, v_2^*, \dots, v_m^*, v_0, \dots, v_m)$$

We now define a well order on $\mathfrak{M}_n(Z)$, $n \geq 0$, by the following recursion.

- (a) Let $u, v \in \mathfrak{M}_0(Z) \setminus \{1\} = S(Z)$. Let $u = u_0 u_1 \cdots u_r$ and $v = v_0 v_1 \cdots v_s$, where $u_0, u_1, \dots, u_r, v_0, v_1, \dots, v_s \in Z$. Then define

$$u \leq_0 v \Leftrightarrow u \leq_{\text{dlex}} v.$$

By Lemma 5.1, \leq_0 is a well order on $\mathfrak{M}_0(Z)$.

- (b) Suppose that a well order \leq_n has been defined on $\mathfrak{M}_n(Z)$ for $n \geq 1$. Let $u, v \in \mathfrak{M}_{n+1}(Z) = M(Z \sqcup [\mathfrak{M}_n(Z)])$. Let $u = u_0[u_1^*]u_1[u_2^*] \cdots [u_r^*]u_r$ and $v = v_0[v_1^*]v_1[v_2^*] \cdots [v_s^*]v_s$, where $u_0, u_1, \dots, u_r, v_0, v_1, \dots, v_s \in M(Z)$ and $u_1^*, u_2^*, \dots, u_r^*, v_1^*, v_2^*, \dots, v_s^* \in \mathfrak{M}_n(Z)$. First suppose that $r = s = m$ for some $m \geq 0$. Then define

$$(41) \quad u \leq_{\text{lex}_{n+1}} v \Leftrightarrow (u_1^*, u_2^*, \dots, u_m^*, u_0, \dots, u_m) \leq_{\text{clex}} (v_1^*, v_2^*, \dots, v_m^*, v_0, \dots, v_m).$$

Since the order \leq_n (resp. \leq_{dlex}) is a well order on $\mathfrak{M}_n(Z)$ by the induction hypothesis (resp. on $M(Z)$), the order \leq_{clex} is a well order on $\mathfrak{M}_n(Z)^m \times M(Z)^{m+1}$ by Lemma 5.4(b), and hence the order $\leq_{\text{lex}_{n+1}}$ is a well order on $\mathfrak{M}_{n+1}^m(Z)$. In general define

$$(42) \quad u \leq_{n+1} v \Leftrightarrow u \leq_{\text{dgp}, \text{brp}, \text{lex}_{n+1}} v \Leftrightarrow \begin{cases} u <_{\text{dgp}} v, \\ \text{or } u =_{\text{dgp}} v \text{ and } u <_{\text{brp}} v, \\ \text{or } u =_{\text{dgp}} v, u =_{\text{brp}} v \text{ and } u \leq_{\text{lex}_{n+1}} v. \end{cases}$$

Since the orders $\leq_{\text{dgp}}, \leq_{\text{brp}}$ satisfy the descending chain condition and $\leq_{\text{lex}_{n+1}}$ is a well order, we conclude that \leq_{n+1} is a well order on $\mathfrak{M}_{n+1}(Z)$ from Lemma 5.4(a).

From the definition of \leq_n for $n \geq 0$, we see that the restriction of the order \leq_{n+1} to $\mathfrak{M}_n(Z)$ equals to the order \leq_n . Thus we can define the order

$$(43) \quad \leq_{\text{db}} := \varinjlim \leq_n = \bigcup \leq_n$$

on the direct system $\mathfrak{M}(Z) = \varinjlim \mathfrak{M}_n(Z)$.

We note that if \leq_{n+1} were defined by $\leq'_{n+1} := \leq_{\text{brp}, \text{lex}_{n+1}}$, then the resulting order $\leq'_{\text{db}} = \varinjlim \leq'_n$ would not be a well order. For example $[u][v] >'_{\text{db}} [[u][v]] >'_{\text{db}} [[[u][v]]] >'_{\text{db}} \cdots$ is an infinite decreasing chain.

Lemma 5.5. *The order \leq_{db} is a well order on $\mathfrak{M}(Z)$.*

Proof. Since \leq_{db} is a linear order on $\mathfrak{M}(Z)$ as a direct limit of linear orders \leq_n , we only need to verify that \leq_{db} satisfies the descending chain condition.

Let a descending chain $v_1 \geq_{\text{db}} v_2 \geq_{\text{db}} \cdots$ in $\mathfrak{M}(Z)$ be given. Since \geq_{dgp} satisfies the descending chain condition, there exists $\ell \geq 0$ such that $v_\ell =_{\text{dgp}} v_{\ell+1} =_{\text{dgp}} \cdots$. Thus $\deg_P(v_\ell) = \deg_P(v_{\ell+1}) = \cdots = k$ for some $k \geq 0$. Then $v_\ell, v_{\ell+1}, \dots$ are in

$$\mathfrak{M}_{(k)} := \{u \in \mathfrak{M}(Z) \mid \deg_P(u) \leq k\}.$$

Note that $\mathfrak{M}_{(k)} \subseteq \mathfrak{M}_k(Z)$. Since the restriction of \leq_{db} to $\mathfrak{M}_k(Z)$ and hence to $\mathfrak{M}_{(k)}$ is \leq_k which, as shown above, satisfies the descending chain condition, the chain $v_\ell \geq_{\text{db}} v_{\ell+1} \geq_{\text{db}} \cdots$ stabilizes after finite steps. Therefore, $v_1 \geq_{\text{db}} v_2 \geq_{\text{db}} \cdots$ stabilizes after finite steps. \square

Definition 5.6. A well order \leq_α on $\mathfrak{M}(Z)$ is called **bracket compatible** (resp. **left (multiplication) compatible**, resp. **right (multiplication) compatible**) if

$$u \leq_\alpha v \Rightarrow [u] \leq_\alpha [v], \text{ (resp. } wu \leq_\alpha wv, \text{ resp. } uw \leq_\alpha vw, \text{ for all } w \in \mathfrak{M}(Z)).$$

Lemma 5.7. A well order \leq is a monomial order on $\mathfrak{M}(Z)$ if and only if \leq is bracket compatible, left compatible and right compatible.

Proof. Suppose that a well order \leq is a monomial order. Let $u, v \in \mathfrak{M}(Z)$ with $u \leq v$. By taking $q = [\star]$, $w\star$ and $\star w$ with $w \in \mathfrak{M}(Z)$, we obtain $[u] \leq [v]$, $wu \leq wv$ and $uw \leq vw$ respectively, proving the bracket compatibility, left compatibility and right compatibility.

Conversely, suppose that a well order \leq is bracket compatible, left compatible and right compatible. Let $u, v \in \mathfrak{M}(Z)$ with $u \leq v$ and let $q \in \mathfrak{M}^*(Z)$ be given. We prove $q|_u \leq q|_v$ by induction on the depth $\text{depth}(q) \geq 0$ of q . If $\text{depth}(q) = 0$, then $q \in M(Z \cup \{\star\})$ and hence $q = w_1 \star w_2$ where $w_1, w_2 \in M(Z)$. Then by the left and right compatibility, we have $q|_u \leq q|_v$. Assume that $q|_u \leq q|_v$ has been proved for $q \in \mathfrak{M}^*(Z)$ with $\text{depth}(q) \leq n$ where $n \geq 0$ and consider $q \in \mathfrak{M}^*(Z)$ with $\text{depth}(q) = n + 1$. If the \star in q is not in a bracket, then $q = w_1 \star w_2$, where $w_1, w_2 \in \mathfrak{M}(Z)$. Then we have $q|_u \leq q|_v$ by the left and right compatibility. If the \star in q is in a bracket, then $q = w_1 [q'] w_2$ with $w_1, w_2 \in \mathfrak{M}(Z)$ and $q' \in \mathfrak{M}^*(Z)$ with $\text{depth}(q') = n$. Hence by the induction hypothesis, we have $q'|_u \leq q'|_v$. Then by bracket, left and right compatibility of \leq , we further have $q|_u \leq q|_v$, completing the induction. \square

Theorem 5.8. The order \leq_{db} is a monomial order on $\mathfrak{M}(Z)$.

Proof. By Lemma 5.5, the order \leq_{db} is a well order on $\mathfrak{M}(Z)$. So we just need to prove that \leq_{db} is bracket compatible, left compatible and right compatible by Lemma 5.7. Let $u, v \in \mathfrak{M}(Z)$. Then there exists a natural number n such that $u, v \in \mathfrak{M}_n(Z)$. Suppose that

$$(44) \quad u = u_0 [u_1^*] u_1 [u_2^*] \cdots [u_r^*] u_r, \quad v = v_0 [v_1^*] v_1 [v_2^*] \cdots [v_s^*] v_s,$$

where $u_0, u_1, \dots, u_r, v_0, v_1, \dots, v_s \in M(Z)$ and $u_1^*, u_2^*, \dots, u_r^*, v_1^*, v_2^*, \dots, v_s^* \in \mathfrak{M}_{n-1}(Z)$. First we prove that \leq_{db} is bracket compatible. By the definition of \leq_{db} , we just need to prove

$$u \leq_n v \Rightarrow [u] \leq_{n+1} [v].$$

Suppose that $u \leq_n v$. By the definition of \leq_n , we have the following three cases.

Case 1 $u <_{\text{dgp}} v$. This means that $\deg_P(u) < \deg_P(v)$. Then we have $\deg_P([u]) = \deg_P(u) + 1 < \deg_P(v) + 1 = \deg_P([v])$. This shows that $[u] \leq_{n+1} [v]$ by the definition of \leq_{n+1} .

Case 2 $u =_{\text{dgp}} v$ and $u <_{\text{brp}} v$. Then we have $[u] =_{\text{dgp}} [v]$. Since the P -breadth of $[u]$ and $[v]$ are equal to 1, we have $[u] =_{\text{brp}} [v]$. Since $u \leq_n v$ and by the definition of lex_{n+1} (that is, by Eq. (41)), $[u] \leq_{\text{lex}_{n+1}} [v]$. Then by Eq. (42), we have $[u] \leq_{n+1} [v]$.

Case 3 $u =_{\text{dgp}} v$, $u =_{\text{brp}} v$ and $u \leq_{\text{lex}_n} v$. This means that $\deg_P(u) = \deg_P(v)$, $r = s$ and $(u_1^*, u_2^*, \dots, u_r^*, u_0, \dots, u_r) \leq_{\text{cllex}} (v_1^*, v_2^*, \dots, v_s^*, v_0, \dots, v_s)$. Then we have $[u] =_{\text{dgp}} [v]$ and $[u] =_{\text{brp}} [v]$. Thus we have $[u] \leq_{n+1} [v]$ since $u \leq_n v$.

Hence the order \leq_{db} is bracket compatible. Next, we prove that \leq_{db} is left compatible. For any $w \in \mathfrak{M}(Z)$, take $m \geq n$ such that $wu, wv \in \mathfrak{M}_m(Z)$. Denote

$$w = w_0 [w_1^*] w_1 [w_2^*] \cdots [w_t^*] w_t,$$

with $w_0, w_1, \dots, w_t \in M(Z)$ and $w_1^*, w_2^*, \dots, w_t^* \in \mathfrak{M}_{m-1}(Z)$. Then with the notation in Eq. (44), we have

$$wu = w_0 [w_1^*] w_1 [w_2^*] \cdots [w_t^*] w_t u_0 [u_1^*] u_1 [u_2^*] \cdots [u_r^*] u_r$$

and

$$wv = w_0[w_1^*]w_1[w_2^*] \cdots [w_t^*]w_tv_0[v_1^*]v_1[v_2^*] \cdots [v_s^*]v_s.$$

Suppose that $u \leq_n v$. To prove $wu \leq_m wv$, we only need to consider the following three cases.

Case 1 $u <_{\text{dgp}} v$. Then we have $\deg_P(wu) = \deg_P(w) + \deg_P(u) < \deg_P(w) + \deg_P(v) = \deg_P(wv)$, and hence $wu <_{\text{dgp}} wv$. Thus we get $wu \leq_m wv$.

Case 2 $u =_{\text{dgp}} v$ and $u <_{\text{brp}} v$. Then we obtain $\deg_P(wu) = \deg_P(wv)$ and $t + r < t + s$. This means that $wu =_{\text{dgp}} wv$ and $wu <_{\text{brp}} wv$, and hence $wu \leq_m wv$.

Case 3 $u =_{\text{dgp}} v$, $u =_{\text{brp}} v$ and $u \leq_{\text{lex}_n} v$. Then we have $wu =_{\text{dgp}} wv$, $wu =_{\text{brp}} wv$ and

$$(u_1^*, u_2^*, \dots, u_r^*, u_0, \dots, u_r) \leq_{\text{clex}} (v_1^*, v_2^*, \dots, v_s^*, v_0, \dots, v_s).$$

Thus we obtain

$$\begin{aligned} & (w_1^*, w_2^*, \dots, w_t^*, u_1^*, u_2^*, \dots, u_r^*, w_0, \dots, w_t u_0, \dots, u_r) \\ & \leq_{\text{clex}} (w_1^*, w_2^*, \dots, w_t^*, v_1^*, v_2^*, \dots, v_s^*, w_0, \dots, w_t v_0, \dots, v_s). \end{aligned}$$

Hence we get $wu \leq_{\text{lex}_m} wv$. Thus we have $wu \leq_m wv$. This completes proof of left compatibility of order \leq_{db} . The proof of the right compatibility is the same, completing the proof. \square

5.2. Consequences on Rota-Baxter type operators. We now verify that all the operators listed in Conjecture 2.37 are Rota-Baxter type operators and obtain the free objects in the corresponding categories of algebras.

Proposition 5.9. *Let $\phi(x, y) = [x][y] - [B(x, y)]$ where $B(x, y)$ is in RBNF and has total operator degree ≤ 1 . More precisely,*

$$\begin{aligned} B(x, y) := & a_0 y[x] + a_1 x[y] + b_0 [y]x + b_1 [x]y + c_0 [yx] \\ (45) \quad & + c_1 [xy] + d_0 x[1]y + d_1 xy[1] + d_2 y[1]x \\ & + d_3 yx[1] + d_4 [1]xy + d_5 [1]yx + \varepsilon_0 yx + \varepsilon_1 xy \end{aligned}$$

where $a_i, b_j, c_k, d_\ell, \varepsilon_t \in \mathbf{k}$ ($0 \leq i, j, k, t \leq 1, 0 \leq \ell \leq 5$). Then Π_ϕ is compatible with the monomial order \leq_{db} in Theorem 5.8.

Proof. For each of the monomials $g(x, y)$ in Eq. (45), we have

$$\deg_P(g) := \deg_P(g(x, y)) \leq 1.$$

Hence for any $u, v \in \mathfrak{M}(Z)$, we have

$$\deg_P([g(u, v)]) = \deg_P(u) + \deg_P(v) + \deg_P(g) + 1 \leq \deg_P(u) + \deg_P(v) + 2 = \deg_P([u][v]).$$

Further, $[g(u, v)] <_{\text{brp}} [u][v]$. Thus $[g(u, v)] <_{\text{db}} [u][v]$, and hence $\overline{[B(u, v)]} <_{\text{db}} [u][v]$. \square

Theorem 5.10. *Let $\phi := [x][y] - [B(x, y)]$, where $B(x, y)$ is in the list in Conjecture 2.37.*

- (a) $B(x, y)$ and the corresponding operator P are of Rota-Baxter type.
- (b) All the statements in Theorem 4.9 hold for $B(x, y)$.
- (c) The free ϕ -algebra on a well ordered set Z has its explicit construction in Theorem 4.10.

In particular, the theorem holds for the Rota-Baxter operator, the Nijenhuis operator and the TD operator.

Proof. (a) First note that all the 14 expressions listed in Conjecture 2.37 are in RBNF. Further by Proposition 5.9, the rewriting systems Π_ϕ from these expressions are compatible with the monomial order \leq_{db} . Hence the rewriting systems are terminating by Theorem 3.10. Thus we only need to check the ϕ -reducibility.

The expressions (e) and (f) are known to have the ϕ -reducibility [14, 18, 31]. The ϕ -reducibility of expressions (a), (b), (m) and (n) are easy to check.

For expression (c): $B(x, y) := x \lfloor y \rfloor + y \lfloor x \rfloor$, we have

$$\begin{aligned} B(B(u, v), w) &= B(u, v) \lfloor w \rfloor + w \lfloor B(u, v) \rfloor \\ &= (u \lfloor v \rfloor + v \lfloor u \rfloor) \lfloor w \rfloor + w \lfloor u \lfloor v \rfloor + v \lfloor u \rfloor \rfloor \\ &\xrightarrow{*}_\phi u \lfloor v \lfloor w \rfloor \rfloor + u \lfloor w \lfloor v \rfloor \rfloor + v \lfloor u \lfloor w \rfloor \rfloor + v \lfloor w \lfloor u \rfloor \rfloor + w \lfloor u \lfloor v \rfloor \rfloor + w \lfloor v \lfloor u \rfloor \rfloor. \end{aligned}$$

and

$$\begin{aligned} B(u, B(v, w)) &= u \lfloor B(v, w) \rfloor + B(v, w) \lfloor u \rfloor \\ &= u \lfloor v \lfloor w \rfloor + w \lfloor v \rfloor \rfloor + (v \lfloor w \rfloor + w \lfloor v \rfloor) \lfloor u \rfloor \\ &\xrightarrow{*}_\phi u \lfloor v \lfloor w \rfloor \rfloor + u \lfloor w \lfloor v \rfloor \rfloor + v \lfloor w \lfloor u \rfloor \rfloor + v \lfloor u \lfloor w \rfloor \rfloor + w \lfloor v \lfloor u \rfloor \rfloor + w \lfloor u \lfloor v \rfloor \rfloor. \end{aligned}$$

Thus $B(B(u, v), w) - B(u, B(v, w)) \xrightarrow{*}_\phi 0$ by Theorem 2.20.

The verification of expression (d) is similar to expression (c).

We next check expression (k). Then the expressions (g), (h) and (i) are subexpressions of expression (k) and can be verified similarly. For expression (k), we have

$$\begin{aligned} B(B(u, v), w) &= B(u, v) \lfloor w \rfloor + \lfloor B(u, v) \rfloor w - B(u, v) \lfloor 1 \rfloor w - \lfloor B(u, v) w \rfloor + \lambda B(u, v) w \\ &\xrightarrow{*}_\phi u \lfloor v \lfloor w \rfloor \rfloor + u \lfloor \lfloor v \rfloor w \rfloor - u \lfloor v \lfloor 1 \rfloor w \rfloor - u \lfloor \lfloor v w \rfloor \rfloor + \lambda u \lfloor v w \rfloor + \lfloor u \rfloor v \lfloor w \rfloor \\ &\quad - u \lfloor 1 \rfloor v \lfloor w \rfloor - \lfloor u v \rfloor w + \lfloor u v \rfloor \lfloor 1 \rfloor w + \lfloor \lfloor u v w \rfloor \rfloor - 2\lambda \lfloor u v w \rfloor + \lambda u v \lfloor w \rfloor \\ &\quad + \lfloor u \rfloor v \lfloor w \rfloor + \lfloor \lfloor u \rfloor v \rfloor w - \lfloor u \lfloor 1 \rfloor v \rfloor w - \lfloor \lfloor u v \rfloor \rfloor w + \lambda \lfloor u v \rfloor w - \lfloor u \rfloor v \lfloor 1 \rfloor w \\ &\quad + u \lfloor 1 \rfloor v \lfloor 1 \rfloor w - \lambda u v \lfloor 1 \rfloor w - \lfloor u \rfloor v \lfloor w \rfloor - \lfloor \lfloor u \rfloor v w \rfloor + \lfloor u \lfloor 1 \rfloor v w \rfloor + \lambda \lfloor u \rfloor v w \\ &\quad - \lambda u \lfloor 1 \rfloor v w + \lambda^2 u v w \end{aligned}$$

and

$$\begin{aligned} B(u, B(v, w)) &= u \lfloor B(v, w) \rfloor + \lfloor u \rfloor B(v, w) - u \lfloor 1 \rfloor B(v, w) - \lfloor u B(v, w) \rfloor + \lambda u B(v, w) \\ &\xrightarrow{*}_\phi u \lfloor v \lfloor w \rfloor \rfloor + u \lfloor \lfloor v \rfloor w \rfloor - u \lfloor v \lfloor 1 \rfloor w \rfloor - u \lfloor \lfloor v w \rfloor \rfloor + \lambda u \lfloor v w \rfloor + \lfloor u \rfloor v \lfloor w \rfloor \\ &\quad + \lfloor u \rfloor v \lfloor w \rfloor + \lfloor \lfloor u \rfloor v \rfloor w - \lfloor u \lfloor 1 \rfloor v \rfloor w - \lfloor \lfloor u v \rfloor \rfloor w + \lambda \lfloor u v \rfloor w - \lfloor u \rfloor v \lfloor 1 \rfloor w \\ &\quad - \lfloor \lfloor u \rfloor v w \rfloor + \lfloor u \lfloor 1 \rfloor v w \rfloor + \lfloor \lfloor u v w \rfloor \rfloor - 2\lambda \lfloor u v w \rfloor + \lambda \lfloor u \rfloor v w - u \lfloor 1 \rfloor v \lfloor w \rfloor \\ &\quad + u \lfloor 1 \rfloor v \lfloor 1 \rfloor w - \lambda u \lfloor 1 \rfloor v w - \lfloor u v \rfloor w - \lfloor u \rfloor v \lfloor w \rfloor + \lfloor u v \rfloor \lfloor 1 \rfloor w + \lambda u v \lfloor w \rfloor \\ &\quad - \lambda u v \lfloor 1 \rfloor w + \lambda^2 u v w. \end{aligned}$$

Now the i -th term in the expansion of $B(B(u, v), w)$ matches with the $\sigma(i)$ -th term in the expansion of $B(u, B(v, w))$. Here the permutation $\sigma \in S_{26}$ is

$$\begin{pmatrix} i \\ \sigma(i) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 1 & 2 & 3 & 4 & 5 & 6 & 18 & 21 & 23 & 15 & 16 & 24 & 7 & 8 \\ 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 \\ 9 & 10 & 11 & 12 & 19 & 25 & 22 & 13 & 14 & 17 & 20 & 26 \end{pmatrix}$$

Thus $B(B(u, v), w) - B(u, B(v, w)) \xrightarrow{*}_{\phi} 0$.

We finally verify expression (l). Then the remaining expression (j), obtained from expression (l) by replacing $[1]xy$ with $xy[1]$, is similarly verified. Expression (l) is $B(x, y) := x[y] + [x]y - x[1]y - [1]xy + \lambda xy$. So we have

$$\begin{aligned} B(B(u, v), w) &= B(u, v)[w] + [B(u, v)]w - B(u, v)[1]w - [1]B(u, v)w + \lambda B(u, v)w \\ &\xrightarrow{*}_{\phi} u[v[w]] + u[[v]w] - u[v[1]w] - u[[1]vw] + \lambda u[vw] + [u]v[w] \\ &\quad - u[1]v[w] - [1]uv[w] + \lambda uv[w] + [u]v]w + [[u]v]w - [u[1]v]w \\ &\quad - [[1]uv]w + \lambda [uv]w - u[[v]]w + u[[1]v]w - [u]v[1]w + u[1]v[1]w \\ &\quad + [1]uv[1]w - \lambda uv[1]w - [1]u[v]w - [[u]]vw + [[1]u]vw + [1]u[1]vw \\ &\quad - \lambda u[1]vw - \lambda [1]uvw + \lambda^2 uvw \end{aligned}$$

and

$$\begin{aligned} B(u, B(v, w)) &= u[B(v, w)] + [u]B(v, w) - u[1]B(v, w) - [1]uB(v, w) + \lambda uB(v, w) \\ &\xrightarrow{*}_{\phi} u[v[w]] + u[[v]w] - u[v[1]w] - u[[1]vw] + \lambda u[vw] + [u]v[w] \\ &\quad + [u]v]w + [[u]v]w - [u[1]v]w - [[1]uv]w + \lambda [uv]w - [u]v[1]w \\ &\quad - [[u]]vw + [[1]u]vw - u[1]v[w] - u[[v]]w + u[[1]v]w + u[1]v[1]w \\ &\quad - [1]uv[w] - [1]u[v]w + [1]uv[1]w + [1]u[1]vw - \lambda [1]uvw + \lambda uv[w] \\ &\quad - \lambda uv[1]w - \lambda u[1]vw + \lambda^2 uvw. \end{aligned}$$

Note that the i -th term in the expansion of $B(B(u, v), w)$ matches with the $\sigma(i)$ -th term in the expansion of $B(u, B(v, w))$. Here the permutation $\sigma \in S_{27}$ is defined by

$$\begin{pmatrix} i \\ \sigma(i) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 1 & 2 & 3 & 4 & 5 & 6 & 15 & 19 & 24 & 7 & 8 & 9 & 10 & 11 \\ 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 & 27 \\ 16 & 17 & 12 & 18 & 21 & 25 & 20 & 13 & 14 & 22 & 26 & 23 & 27 \end{pmatrix}$$

Thus $B(B(u, v), w) - B(u, B(v, w)) \xrightarrow{*}_{\phi} 0$.

(b) follows from Item (a), Corollary 3.13 and Proposition 5.9.

(c) follows from Item (b) and Theorem 4.10. □

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